MATH 550 PDEs Notes

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1 Introduction

This section will survey the principal theoretical issues regarding solvability of PDEs.

1.1 Partial Differential Equations

We define a PDE is the equation takes the following form.

Definition 1.1. ***PDE:*** *An equation of the form,*

$$F(D^k u, D^{k-1}u, ..., Du, u, x) = 0, \tag{1.1}$$

is called a k-th order partial differential equation where $u(x) : U \to \mathbb{R}$ where $x \in U \subset \mathbb{R}^n$ and

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}.$$

Here u is the unknown function and finding all u such that (1.1) is varified qualifies as solving the PDE.

We ideally would like to find a simple expression for **u**. This is in general very difficult or even impossible. Hence the second best we can hope for is to prove the existence of such solutions and possibly their behaviour.

Definition 1.2. ***Linear PDE:*** *has the form,*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x). \tag{1.2}$$

We call the,

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u,$$

the *principle part*.

Definition 1.3. ***Semi-linear PDE:*** *has the form,*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + G(D^{k-1}u, ..., Du, u, x) = 0. \tag{1.3}$$

Here the function G could be non-linear, whereas the principle part is linear.

Definition 1.4. ***Quasi-linear PDE:*** *has the form,*

$$\sum_{|\alpha| = k} F(D^{k-1}u, ..., Du, u, x) D^\alpha u + G(D^{k-1}u, ..., Du, u, x) = 0, \tag{1.4}$$

where F and G could be non-linear functions.

Definition 1.5. ***Fully non-linear PDE:*** *If the PDE is not of the forms* (1.2),(1.3),(1.4).

The same classifications hold if instead the unknown function u is vector valued i.e. $\mathbf{u} : U \to \mathbb{R}^m$ and $\mathbf{u} = (u^1, ..., u^m)$.

1.2 Philosophy of the analysis of PDEs

There is in no proof of a general theory of how to solve PDEs. This theory probably does not exist since PDEs arise from a wide variety of disciplines in the sciences as well as pure mathematics.

Hence, the best way to study PDEs is to study the specific equation at hand and understand its origins and how its solutions may behave.

1.2.1 Well-posed problems and classical solutions

We say a PDE is well posed if,

- A solution to the PDE exists.
- The solution is unique.
- The solution depends continuously on intial data.

By the last point we mean that a small change in the intial data results in a small change in the solution. This condition is of specific importance when the PDE in question arises from physics.

The definition of well-posed-ness does not take into account smoothness of solutions. Obviously, we would at least want a solution to a kth order PDE to be at least k times differentiable. We call such a solution to a PDE a **classical solution**.

1.2.2 Weak solutions and regularity

It turns out that in general seeking classical solutions for all types of PDE is an overly restrictive criteria.

In other words there are numerous well-posed PDE that do not admit a classical solution. For example the scalar conservation law,

$$u_t + F(u)_x = 0. \tag{1.5}$$

These PDE are only well-posed if we allow for generalized or **weak solutions**. By this we mean solutions that may not be smooth or even continuous.

A very reasonable strategy to solve a PDE is to first define an appropriately wide range of solutions and then consider separate the issues of existence and smoothness or regularity of solutions.

1.2.3 General Take-away

I place this section here as it does not belong in any particular section of these notes. Rather it describes general takeaways from learning PDEs and their analysis. This section will make more sense by re-visiting upon reading through the notes.

One of the primary difficulties I found while taking the course and learning through Evan's PDE was the plethora of techniques one could use. Initially, I found there to be a dichotomy between energy methods and maximum principle methods for the canonical linear PDEs. I found that I was confused as to which I should use to prove various claims. The general take-away that I found was that; when we must prove something point-wise about solutions, max principle methods are probably the best. On the other hand, when we must prove something about the integrals of solutions, energy methods will most likely be needed.

1.3 Mollifiers

Before we get into the study of PDEs it is worth going through this brief interlude regarding *mollifier.* At their heart, mollification can be viewed as a way of approximating any function with smooth functions. This technique will be very useful in proving solutions to PDEs.

More precisely, given a function f is like the saw tooth function, we want to find a smooth approximation for f by some function f^ϵ such that as $\epsilon \to 0$, $f^\epsilon \to f$.

Let,

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{on } B(0,1) \\ 0 & \text{in } \mathbb{R}^n \setminus B(0,1). \end{cases} \tag{1.6}$$

We pick C such that $\int \eta \, dx = 1$ i.e. has integral unity over the entire domain. The main idea is that we would like to turn η into a Dirac-delta distribution. We do this by introducing a parameter ϵ. So let,

$$\eta^\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right). \tag{1.7}$$

Notice, the support of η is given by $|x| < 1$ and the support of η^ϵ is given by $|\frac{x}{\epsilon}| < 1$ and so $|x| < \epsilon$.

Now given some f, the *mollification* of f is given as,

$$f^\epsilon = f * \eta^\epsilon = \int_{\mathbb{R}^n} \eta^\epsilon(y) f(x-y)\, dy. \tag{1.8}$$

Some properties of mollification. $f^\epsilon \in C^\infty$ for all $\epsilon > 0$.

Also the mollification converges to f almost everywhere. This can be argued at least heuristically since as $\epsilon \to 0$, $\eta^\epsilon \to \delta_0$ and so $\eta^\epsilon * f \to \delta_0 * f = f(x)$.

2 Four Important Linear PDEs

Here we look at the four fundamental linear PDEs. These equations have known formulas for solutions.

2.1 Transport equation

One of the simplest PDEs is the transport equation, also known as the convection equation,

$$u_t + b \cdot Du = 0, \tag{2.1}$$

in $\mathbb{R}^n \times (0,\infty)$. Here $b \in \mathbb{R}^n$ is a constant vector and $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$.

Let us suppose that there exists a smooth solution to (2.1) u. The key insight here is to recognize that (2.1) asserts that there exists a directional derivative of u that vanishes. Re-interpreting (2.1) as the dot product between,

$$(1,b) \cdot (u_t, Du) = 0,$$

we have that $(1,b) \perp (u_t, Du)$.

Then, fix a point $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and define,

$$z(s) := u(x+sb, t+s).$$

Taking the derivative, we get,

$$\frac{dz}{ds} = Du(x+sb,t+s) \cdot b + u_t(x+sb,t+s) = 0,$$

following an application of the chain rule. The equation tells us that $z(s)$ must be a constant function in s. Hence, for each point (x, t), u is constant on the line through (x, t) in the direction $(b, 1) \in \mathbb{R}^{n+1}$. So, if we know the value of u on any point on such a line, we know its value everywhere on the domain of definition of (2.1).

2.1.1 Initial-value problem

Consider then the intial value problem,

$$\begin{aligned} u_t + b \cdot Du &= 0 \\ u(x, 0) &= g(x, t). \end{aligned} \tag{2.2}$$

From the previous insight, we know $z(s)$ is constant on the line through (x, t) in the direction $(\mathbf{b}, 1)$ and is represented parametrically as $(x+sb, t+s)$. We know this line intersects the plane $\mathbb{R}^n \times \{t = 0\}$ when $s = -t$, at the point $(x - tb, 0)$. Since u is constant on the line and by the intial condition, $u(x - tb, 0) = g(x - tb)$, the explicit solution is given as,

$$u(x, t) = g(x - tb). \tag{2.3}$$

Theorem 2.1. *Let u_0 be an initial condition for* (2.2) *and let $u_0 \in C^n$. Then,* (2.4) *has a unique solution such that $u \in C^n(\mathbb{R}^n \times (0, \infty])$ and the solution is given by,*

$$u(x, t) = u_0(x - tb).$$

2.2 Non-homogeneous problem

Consider the non-homogeneous counterpart to the problem,

$$\begin{aligned} u_t + \mathbf{b} \cdot Du &= f(x, t) \\ u(x, 0) &= g(x, t). \end{aligned} \tag{2.4}$$

Applying the same insight as before, we find that,

$$\frac{dz}{ds} = Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) = f(x + sb, t + s).$$

Notice then that,

$$\begin{aligned} \int_{-t}^{0} \frac{dz}{ds}\, ds &= z(0) - z(-t) \\ &= u(x, t) - g(x - tb) \\ &= \int_{-t}^{0} f(x + sb, t + s)\, ds \\ &= \int_{0}^{t} f(x + (s - t)b, s)\, ds. \end{aligned}$$

It then follows that,

$$u(x, t) = g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s)\, ds. \tag{2.5}$$

Notice that we derived the solution forms (2.3) and (2.5) by converting the PDE into an ODE. This is a special case of the method of characteristics.

Here a few cautinary remarks and listing or properties are in order.

Consider first the specific example,

$$\begin{cases} 3u_x + 2u_y &= 0 \\ u(x, \frac{2}{3}x) &= \exp(x). \end{cases}$$

Applying Theorem 2.1 directly would be incorrect as the problem is ill-posed. This is since the initial condition lies on a characteristic curve and is hence parallel to any solution u. Therefore, the problem has no solution. It is thus of paramount importance to note that (2.2) and (2.4) are well-posed if the initial condition do not lie on a characteristic curve.

Properties of the Transport equation,

- *Propagation of regularity:* If $u_0 \in C^k$ then $u \in C^k$.
- *Finite speed of propagation:* If u_0 vanishes outside $B(x_0, r)$, then u vanishes outside $B(x_0 + bt, r)$.
- *Propagation of singularity:* If $u_0 \in C^n$ and not in C^{n+1}, then $u \in C^n$ and not in C^{n+1}.

2.3 Classification of Second Order PDEs

Every second order PDE (in two variables) can be written in general or standard form as follows,

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0. \tag{2.6}$$

Theorem 2.2. *By linear transformation of the independent variables* (2.6) *can be reduced to one of the three following forms,*

- ***Elliptic:*** $u_{xx} + u_{yy} +$ *lower order terms* $= 0$ *if* $a_{12}^2 < a_{11}a_{22}$.
- ***Hyperbolic:*** $u_{xx} - u_{yy} +$ *lower order terms* $= 0$ *if* $a_{12}^2 > a_{11}a_{22}$.
- ***Parabolic:*** $u_{xx} +$ *lower order terms* $= 0$ *if* $a_{12}^2 = a_{11}a_{22}$.

2.4 Laplace's equation

The most important of all PDEs is Laplace's equation,

$$\Delta u = 0, \tag{2.7}$$

and its non-homogeneous counterpart, Poisson's equation,

$$-\Delta u = f. \tag{2.8}$$

Here $x \in U \subset \mathbb{R}^n$ and $u(x) : U \to \mathbb{R}$ is the unknown. In (2.8) $f : U \to \mathbb{R}$ is also given.

Definition 2.3. ***Harmonic Function:*** *A* C^2 *function* u *satisfying* (2.7) *is called harmonic.*

2.4.1 Physical interpretation

In Laplace's equation u usually denotes a particular quantity at equilibrium within a system.

Let $U \subset \mathbb{R}^n$ and let $V \subset U$, then it must be the case that the flux of u through ∂V is zero.

$$\int_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} = 0.$$

Here $\mathbf{F}$ denotes the flux density and $\hat{\mathbf{n}}$ is the unit outward normal with respect to ∂V. Application of the divergence theorem yeilds,

$$\int_V \operatorname{div} \mathbf{F} = \int_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} = 0.$$

Therefore it must be the case that div$\mathbf{F} = 0$. In many physical scenarios the flux is given by the gradient of the quantity u,

$$\mathbf{F} = -aDu.$$

$a \in \mathbb{R}^+$ and the negative sign denotes a custom to show any physically relevant quantities (temperature, chemical concentration etc.) move from higher to lower concentrations.

Hence we arrive at Laplace's equation,

$$\text{div}\,(Du) = \Delta u.$$

2.4.2 Fundamental Solution

We now derive the fundamental solution for Laplace's equation. We observe first that (2.7) is rotationally invariant.

Theorem 2.4. ***Rotational invariance:*** *Laplace's equation is rotationally invariant. Explicitly, suppose u is harmonic and $v(x) = u(Rx)$ where R is some orthogonal matrix, then v is also harmonic.*

Proof. □

Hence, we search for radial functions i.e. $u(x) = v(r)$ where $|x| = \sqrt{x_1^2 + ... + x_n^2} = r$.

Using the chain rule we have that,

$$\frac{\partial r}{\partial x_i} = (x_1^2 + ... + x_n^2)^{-1/2} x_i = \frac{x_i}{r}.$$

It follows from the fact that $u(x) = v(r)$,

$$\partial_{x_i} u(x) = v'(r)\frac{x_i}{r},$$

and

$$u_{x_i x_i} = v'' \frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

Notice then, this is the case for the i-th component of the Laplacian. Summing over we gain,

$$\begin{aligned}\Delta u &= \sum_{i=1}^{n} u_{x_i x_i} \\ &= \frac{v''}{r^2}\sum_{i=1}^{n} x_i^2 + \frac{v'}{r}\sum_{i=1}^{n} 1 - \frac{v'}{r^3}\sum_{i=1}^{n} x_i^2 \\ &= v'' + \frac{n-1}{r}v' = 0.\end{aligned}$$

Notice, we have transformed (2.7) into a second order separable ODE.

Upon solving the ODE and taking into account certain cases, we get that,

$$v(r) = \begin{cases} b \ln r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3). \end{cases}$$

Definition 2.5. *The function,*

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \ln |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3), \end{cases} \tag{2.9}$$

is the fundamental solution of Laplace's equation.

Here $\alpha(n)$ denotes the volume of the unit ball.

2.4.3 Poisson's Equation

We now use the fundamental solution to Laplace's equation to solve Poisson's equation.

Notice, the mapping $x \mapsto \Phi(x)$ is harmonic for $x \neq 0$. Furthermore, shifting the origin to a new point, the PDE is unchanged i.e. $x \mapsto \Phi(x-y)$ for $x \neq y$. If we then let $f : \mathbb{R}^n \to \mathbb{R}$ and not that $x \mapsto \Phi(x-y)f(y)$ for $x \neq y$ is also harmonic for each $y \in \mathbb{R}^n$. Furthermore, so is the sum of finitely many such y.

Warning: This may suggest the convolution $\Phi * f$ is harmonic. However this is not true as we do not have any regularity conditions that justify bringing the Laplacian under the integral sign. In particular, there exists a singularity at $x = y$.

Let f be twice continuously differentiable with compact support.

Theorem 2.6. ***Solving Poisson's Equation:*** *Let $u = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy$. Then,*

- $u \in C^2$
- $-\Delta u = f$ *in* R^n.

Proof. By the symmetry property of convolutions,

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\, dy = \int_{\mathbb{R}^n} \Phi(y)f(x-y)\, dy.$$

Now, due to regularity constraints we cannot naively pass the Laplacian under the integral sign and differentiate with respect to x. However, by the dominated convergence theorem, we are allowed to pass limits under the integral sign. We may do so using the difference quotient.

Let $h \neq 0$ and let $e_i = (0, ..., 1, ..., 0)$, where 1 is the ith enumeration. Then we have that

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(x) \left[\frac{f(x+he_i-y) - f(x-y)}{h} \right] dy.$$

In the limit, as $h \to 0$, the previous equation becomes,

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \Phi(x) \frac{\partial f}{\partial x_i}(x-y)\, dy,$$

uniformly on $\mathbb{R}^n$.

Since $f \in C_c^2$, applying the same procedure again we have,

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} \Phi(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\, dy.$$

This shows the first claim that $u \in C^2(\mathbb{R}^n)$.

Now we prove the second point. We must take care of the singularity at the origin where Φ blows up. This will require some surgery. Fix $\epsilon > 0$ and let $B(0, \epsilon) = B$ denote the ball centred at the origin with radius ϵ. Then, the previous equation can be re-written as two integrals,

$$\begin{aligned}\Delta u(x) &= \int_B \Phi(x)\Delta f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B} \Phi(x)\Delta f(x-y)\, dy \\ &=: I_\epsilon + J_\epsilon.\end{aligned}$$

The integrals are equivalent since $\int_{\mathbb{R}^n} = \int_{\mathbb{R}^n \setminus \{x\}}$ in the Lebesgue sense.

We now examine each of the functionals separately. Consider first I_ϵ. By the triangle inequality we gain the following result,

$$|I_\epsilon| \leq \int_B |\Phi(x)||\Delta f(x-y)|\, dy \leq C \int_B |\Phi(x)|\, dy.$$

The last inequality stems for the fact that f is twice continuously differentiable with compact support and hence is bounded as well as all its derivatives i.e. $|\Delta f(x-y)| \leq C$.

Now, the last integral still is in danger of blowing up. However this will not be the case. Evaluating the integral in polar coordinates we get,

$$\int_B |\Phi(x)|\, dy = \int_0^\epsilon \left(\int_{\partial B} \Phi(y)\, d\sigma \right)\, dr.$$

Here $d\sigma$ is the surface measure i.e. $rd\theta$ is elementary polar coordinate notation. We know that Φ within the integrand will take the form $\Phi = \frac{C}{r^{n-2}}$. Hence the previous integral becomes,

$$\begin{aligned} \int_0^\epsilon \left(\int_{\partial B} \Phi(y)\, d\sigma \right)\, dr &= \int_0^\epsilon \frac{C}{r^{n-2}} |\partial B|\, dr \\ &= \int_0^\epsilon \frac{C}{r^{n-2}} n r^{n-1} \alpha(n)\, dr \\ &= \frac{C\epsilon^2}{2}. \end{aligned}$$

Clearly, let $\epsilon \to 0$ implies $I_\epsilon \to 0$.

Now consider J_ϵ. Integrating by parts,

$$\begin{aligned} J_\epsilon &= \int_{\mathbb{R}^n \setminus B} \Phi(x) \Delta f(x-y)\, dy \\ &= \int_{\partial B} \Phi(y) \frac{\partial f}{\partial \hat{\mathbf{n}}}(x-y)\, dy - \int_{\mathbb{R}^n \setminus B} D\Phi(y) \cdot Df(x-y)\, dy \\ &=: L_\epsilon + K_\epsilon. \end{aligned}$$

A point to note here, we have taken the Laplacian with respect to y, however, before this we took it with respect to x. This is justified since $\Delta_x f = \Delta_y f$.

Similar to before; we consider the functionals L_ϵ, K_ϵ separately. Using the same bounding argument as I_ϵ; L_ϵ is bounded above by,

$$|L_\epsilon| \leq C \int_{\partial B} |\Phi(y)|\, d\sigma(y).$$

Integrating in polar coordinates,

$$C \int_{\partial B} |\Phi(y)|\, d\sigma(y) = \frac{C}{\epsilon^{n-2}} n \epsilon^{n-1} = C\epsilon.$$

Clearly, $\lim_{\epsilon \to 0} L_\epsilon = 0$.

Integrating K_ϵ by parts we have,

$$K_\epsilon = -\int_{\mathbb{R}^n \setminus B} \Delta\Phi(y) f(x-y)\, dy - \int_{\partial B} \frac{\partial \phi}{\partial \hat{\mathbf{n}}}(y) f(x-y)\, dy = -\int_{\partial B} \frac{\partial \phi}{\partial \hat{\mathbf{n}}}(y) f(x-y)\, dy.$$

Then we have that $\frac{\partial \Phi}{\partial \hat{\mathbf{n}}} = D\Phi \cdot \hat{\mathbf{n}}$, where,

$$\begin{aligned} D\Phi &= -\frac{-1}{n\alpha(n)} \frac{y}{|y|^n} \\ \hat{\mathbf{n}} &= \frac{-y}{|y|}. \end{aligned}$$

Notice the negative sign on the normal vector, this is since this is an inward pointing normal due ot the fact that we isolate the singularity around 0. Also, since the ball has radius ϵ it must be the case that $|y| = \epsilon$.

$$K_\epsilon = -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B} f(x-y)\, d\sigma(y),$$

which upon a change of variables brought on by shift the ball to any x yields,

$$\fint_{\partial B(x,\epsilon)} f(y)\, d\sigma(y) \to -f(x), \quad \epsilon \to 0.$$

Hence, $\Delta u = -f(x)$. □

2.5 Mean Value Formulas

We now derive important properties of Laplace's equation, namely; the mean value properties.

Let $U \subset \mathbb{R}^n$ be an open subset and let $u(x)$ be harmonic on U. We want to show that u is equal to its own average on both the ball centred at any $x \in U$ with radius r and the average on the surface of such a ball.

Theorem 2.7. ***Mean Value Property:*** *Let $u \in C^2$ be harmonic on U. Then,*

$$u(x) = \fint_{\partial B(x,r)} u\, dS = \fint_{B(x,r)} u\, dB, \tag{2.10}$$

for all $B(x,r) \subset U$.

Proof. Let,

$$\phi(r) = \fint_{\partial B(x,\epsilon)} u(y)\, dS(y).$$

We want to show that $\phi(r) = u(x)$. Also note, that the surface area of the sphere i.e. $|\partial B|$ is exactly the derivative of the volume of the same sphere. In n dimensions,

$$|\partial B| = nr^{n-1}\alpha(n).$$

Here $\alpha(n) = |B(0,1)|$Then, we may expand ϕ,

$$\phi(r) = \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(x,\epsilon)} u(u)\, dS(y).$$

We would like to differentiate $\phi(r)$, however the integral depends on r. This requires a change of variables. Let $y = x + rz$, so $z = \frac{y-x}{r}$. Notice this, this shifts the ball $B(x,r)$ to $B(0,1)$. Putting this together, we may rewrite the integral in the new variables,

$$\phi(r) = \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(0,1)} u(x+rz) r^{n-1}\, dS(z) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\, dS(z)$$

Now let us differentiate. Using the chain rule, we have,

$$\phi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x+rz) \cdot z\, dS(z) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x+rz) \cdot \frac{x-y}{r}\, dS(z).$$

Now, going back to the original variables,

$$\phi'(r) = \frac{1}{nr^{n-1}\alpha(n)} \int_{\partial B(x,r)} Du(y) \cdot \frac{x-y}{r}\, dS(y).$$

Notice, $\frac{x-y}{r}$ is exactly the unit normal to the surface and is hence $\hat{\mathbf{n}}$. So, $Du \cdot \hat{\mathbf{n}} = \frac{\partial u}{\partial \hat{\mathbf{n}}}$. Then integral can then be written as,

$$\phi'(r) = \fint_{\partial B(x,r)} \frac{\partial u}{\partial \hat{\mathbf{n}}} \, dS(y).$$

Then, applying the divergence theorem, and recalling u is harmonic i.e. $D \cdot (Du) = \Delta u = 0$,

$$\phi'(r) = \fint_{\partial B(x,r)} \frac{\partial u}{\partial \hat{\mathbf{n}}} \, dS(y) = \fint_{\partial B(x,r)} \Delta u \, dS(y) = 0.$$

It follows $\phi(r)$ is constant. So,

$$\lim_{r \to 0} \phi(r) = \lim_{r \to 0} \fint_{\partial B(x,\epsilon)} u(y) \, dS(y) = u(x).$$

To conclude the proof, notice the volume integral can be computed using polar coordinates,

$$\begin{aligned} \fint_{B(x,r)} u(y) \, dy &= \int_0^r \left(\int_{\partial B(x,r)} u \, dS \right) \, ds \\ &= u(x) \int_0^r n\alpha(n) s^{n-1} ds = \alpha(n) r^n u(x). \end{aligned}$$

□

Theorem 2.8. ***Converse to mean value property:*** *If $u \in C^2(U)$ and*

$$u(x) = \fint_{\partial B(x,r)} u(y) \, dS(y),$$

for all $B(x,r) \subset U$, then u is harmonic.

Proof. Assume for a contradiction, that u is not harmonic i.e. $u \neq 0$. Then, without loss of generality, we may assume there exists some $B(x,r) \subset U$ where $\Delta u > 0$ in $B(x,r)$. But this would mean that,

$$0 = \phi'(r) > 0,$$

which is absurd! □

We now consider other important properties of Harmonic functions.

Theorem 2.9. ***Strong Maximum Principle:*** *Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic within you. Then,*

- $\max_{\bar{U}} u = \max_{\partial U} u$.
- *If U is connected and there exists a point $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$, then u is constant within U*

The first point is called the *maximum principle* for Laplace's equation and the second is the *strong maximum principle.*

Proof. Suppose there exists $x_0 \in U$ such that $u(x_0) = M := \max_{\bar{U}} u$. Then, by the mean value property, for any ball $B(x_0, r)$ such that $0 < r < d(x_0, \partial U)$,

$$M = \fint_{B(x_0,r)} u(y) \, dy \leq M.$$

Equality holds when $u \equiv M$ within $B(x_0, r)$. So, $u(y) = M$ for all $y \in B(x_0, r)$. Hence the set is $\{x \in U : u(x) = M\}$ is both open and relatively closed in U. Hence the closure equals U if U is connected. This proves the strong max principle from which the max principle follows. □

We will now look towards proving regularity of harmonic functions. In other words, we know harmonic $u \in C^2$ and we will now prove that this necessarily means that $u \in C^\infty$. That is harmonic functions are automatically infinitely smooth.

Theorem 2.10. ***Smoothness of Harmonic Functions:*** *If $u \in C(U)$ and satisfies the mean value property for each $B(x,r) \subset U$, then $u \in C^\infty(U)$.*

Note that by the converse of the mean value property, u is harmonic. Also u need not be smooth or even continuous up to ∂U.

Proof. Let $\eta \in C_c^\infty$ and let $\int_{\mathbb{R}}^n \eta = 1$. Moreover, upon rescaling it follows that, $\frac{1}{\epsilon}\eta\left(\frac{r}{\epsilon}\right)$ will have compact support within $B(0,\epsilon)$. Notice, η is the standard mollifier.

Let $u_\epsilon = u * \eta_\epsilon$ in $U_\epsilon = \{x \in U : d(x, \partial U) > \epsilon\}$. Now, since the derivative can pass onto any one of the functions within the convolution, it follows that u_ϵ is infinitely smooth since η is a mollifier and hence smooth.

We will now prove u is smooth by showing that $u = u_\epsilon$ on U_ϵ. If $x \in U_\epsilon$,

$$\begin{aligned}
u_\epsilon(x) &= \int_U \eta_\epsilon(x-y)u(y)\,dy \\
&= \frac{1}{\epsilon^n}\int_{B(0,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right) u(y)\,dy \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)\left(\int_{\partial B(x,r)} u\,dS\right)\,dr \\
&= u(x)\frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) n\alpha(n)r^{n-1}\,dr \\
&= u(x)\int_{B(0,\epsilon)} \eta_\epsilon\,dy = u(x).
\end{aligned}$$

The second line follows from a change of variables and being constrained to the ϵ-ball. We then used polar coordinates and the last line followed from using the fact that the mollifier has mass 1 over its support.

Therefore it is clear that $u_\epsilon = u$ on the set U_ϵ and hence u is smooth for any $\epsilon > 0$. □

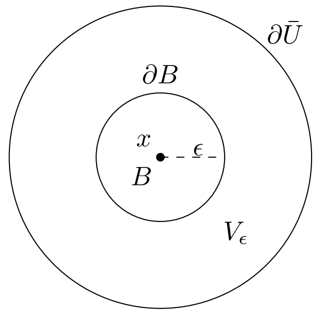
2.5.1 Green's Functions

We now want to find a general representation theorem for Poisson's equation on the open, bounded set U. To be precise we have the boundary value problem,

$$\begin{cases} -\Delta u &= f \\ u|_{\partial U} &= g \end{cases} \tag{2.11}$$

We now derive Green's function for this setup.

Let $u \in C^2(\bar{U})$ and let $x \in U$ and $\epsilon > 0$ be such that $B(x,\epsilon) \subset U$. Now consider the modified domain $V_\epsilon := U \setminus B(x,\epsilon)$. Without loss of generality, we sketch the domain.



We then constrain $u(y)$ and $\Phi(y-x))$ to be on V_ϵ. Then, computing,

$$\int_{V_\epsilon} u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\, dy = \int_{\partial V_\epsilon} u(y)\frac{\partial \Phi}{\partial \hat{\mathbf{n}}}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \hat{\mathbf{n}}}\, dS(y). \tag{2.12}$$

The idea here is that we want to use Green's second identity. Furthermore, we do not want to integrate the singularity and hence we must integrate over V_ϵ. Notice also the non-boundary integral with $Du \cdot D\Phi$ will be zero as we will have 2 copies of the term subtracted.

Here $\hat{\mathbf{n}}$ is the outer normal on V_ϵ Recall however, this set has two boundaries, namely ∂U and $\partial B(x,\epsilon)$. And so the boundary integral in (2.12) must be decomposed,

$$\int_{\partial V_\epsilon} u(y)\frac{\partial \Phi}{\partial \hat{\mathbf{n}}}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \hat{\mathbf{n}}}\, dS(y) = \left(\int_{\partial U} + \int_{\partial B(0,\epsilon)}\right) u(y)\frac{\partial \Phi}{\partial \hat{\mathbf{n}}}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \hat{\mathbf{n}}}\, dS(y).$$

Keeping this in mind; since Φ is harmonic, (2.12) simplifies to,

$$-\int_{V_\epsilon} \Phi(y-x)\Delta u(y)\, dy = \int_{\partial V_\epsilon} u(y)\frac{\partial \Phi}{\partial \hat{\mathbf{n}}}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \hat{\mathbf{n}}}\, dS(y).$$

Consider then the integral over $\partial B(0,\epsilon)$. Since $u \in C^2(\bar{U})$, u and its derivatives are bounded and hence,

$$|\int_{\partial B(0,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial \hat{\mathbf{n}}}\, dS(y)| \leq \sup_{y\in\bar{U}} |Du|\epsilon^{n-1} \max_{\partial B(0,\epsilon)} |\Phi|.$$

Taking $\epsilon \to 0$, clearly this term becomes negligibly small. So we gain a further simplification,

$$-\int_{V_\epsilon} \Phi(y-x)\Delta u(y)\, dy = \int_{\partial V_\epsilon} u(y)\frac{\partial \Phi}{\partial \hat{\mathbf{n}}}(y-x)\, dS(y).$$

Now, recall that,

$$\int_{\partial B(x,\epsilon)} u(y)\frac{\partial \Phi}{\partial \hat{\mathbf{n}}}(y-x)\, dS(y) = \fint_{\partial B(x,\epsilon)} u(y)\, dS(y) \to u(x),$$

as $\epsilon \to 0$.

Then, by sending $\epsilon \to 0$ in (2.12), we gain,

$$u(x) = \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \hat{\mathbf{n}}}(y) - u(y)\frac{\partial \Phi}{\partial \hat{\mathbf{n}}}(y-x)\, dS(y) - \int_U \Phi(y-x)\Delta u(y)\, dy. \tag{2.13}$$

Here the identity is true for any $x \in U$ and $u \in C^2(U)$.

Now, consider (2.13); in the case of the BVP concerning Poisson's equation, we know the value of u on the boundary i.e. $u = g$. Also by the PDE, $-\Delta u = f$. We do not however know anything about $\frac{\partial u}{\partial \hat{\mathbf{n}}}$.

To figure out the normal derivative, we will need a *corrector function.* Let $\Phi_x(y)$ be this function. If it exists, we want it to be,

$$\begin{cases} \Delta\Phi_x(y) & = 0 \text{ in } \Omega \\ \Phi_x(y) & = \Phi(y-x) \text{ on } \partial\Omega. \end{cases}$$

In words, the corrector function is harmonic and on the boundary it is simply the fundamental solution shifted by x amount.

Warning: We are not claiming that this corrector function exists. Indeed, in some cases it does not. But, if it exists, observe the following.

Consider the following integral,

$$\int_\Omega \Phi_x(y)\Delta u(y)\,dy = \int_{\partial\Omega} \Phi_x(y)\frac{\partial u}{\partial \hat{\mathbf{n}}}\,dS(y) - \int_\Omega D\Phi_x(y)\cdot Du\,dy.$$

Since we know information about $\Delta\Phi_x(y) = 0$, let us integrate the integral over Ω by parts again,

$$\int_\Omega \Phi_x(y)\Delta u(y)\,dy = \int_{\partial\Omega} \Phi_x(y)\frac{\partial u}{\partial \hat{\mathbf{n}}}\,dS(y) - \int_{\partial\Omega} \frac{\partial\Phi_x(y)}{\partial \hat{\mathbf{n}}}u(y) + \underbrace{\int_\Omega \Delta\Phi_x(y)u(y)\,dy}_{=0}.$$

Recall that, $\Phi_x(y) = \Phi(x-y)$ on the boundary. Hence,

$$\int_\Omega \Phi_x(y)\Delta u(y)\,dy = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \hat{\mathbf{n}}} - \frac{\partial\Phi_x(y)}{\partial \hat{\mathbf{n}}}u(y)\,dS(y). \tag{2.14}$$

Now we may combine (2.13) and (2.14) to gain a convenient simplification. Writing these two equations out explicitly,

$$\begin{aligned} u(x) &= \int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \hat{\mathbf{n}}}(y) - u(y)\frac{\partial\Phi}{\partial \hat{\mathbf{n}}}(y-x)\,dS(y) - \int_\Omega \Phi(y-x)\Delta u(y)\,dy, \\ 0 &= \int_{\partial\Omega} \Phi(y-x)\frac{\partial u(y)}{\partial \hat{\mathbf{n}}} - \frac{\partial\Phi_x(y)}{\partial \hat{\mathbf{n}}}u(y)\,dS(y) - \int_\Omega \Phi_x(y)\Delta u(y)\,dy. \end{aligned}$$

Then subtracting the two, we gain,

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial}{\partial \hat{\mathbf{n}}}\left(\Phi(y-x) - \Phi_x(y)\right)\,dS(y) - \int_\Omega \left(\Phi(y-x) - \Phi_x(y)\right)\,dy. \tag{2.15}$$

Definition 2.11. ***Green's Function:*** *Considering* (2.15)*, we define,*

$$G(x,y) := \Phi(y-x) - \Phi_x(y). \tag{2.16}$$

Then, (2.15) provides us the representation solution, $u(x)$, in terms of Green's function for the BVP governed by Poisson's equation. Furthermore, in this case (2.15) can be explicitly stated, since we know $-\Delta u = f$ and $u = g$ on the boundary.

In general, computing Green's functions for exotic boundary geometries is very difficult.

We will now compute Poisson's formula for the ball, using Green's functions.

Poisson's formula for the ball

Consider $B(0,1)$ open, we have that u on the ball is harmonic, also $u = g$ on $\partial B(0,1)$. We will now solve this BVP using Green's functions.

Let $x \in B$ be fixed. We wan to find a corrector function $\Phi_x = \Phi_x(y)$ such that,

$$\begin{cases} -\Delta \Phi_x &= 0 \text{ in } B \\ \Phi_x(y) &= \Phi(y-x) \text{ on } \partial B \end{cases}$$

We will find an explicit formula for Φ_x.

Recall that $\Phi(y-x)$ will have a singularity at x. To avoid this we will use a reflection i.e. $x \mapsto \bar{x}$. We may do this by,

$$|\bar{x}| = \frac{|x|}{|x|^2} = \frac{1}{|x|} > 1.$$

The reason it is greater than 1 is since x is in the unit ball. Hence we are guaranteed that $\bar{x}$ is outside the unit ball.

So, let us define the corrector function,

$$\Phi_x = \Phi(|x|(y-\bar{x})). \tag{2.17}$$

We now treat (2.17) as an ansatz to the BVP on Φ_x. Checking (2.17) is harmonic is essentially an exercise in the chain rule.

It is harder to show that if $y \in \partial B$ then $\phi_x(y) = \Phi(y-x)$. Recall that,

$$.\Phi(|x|(y-\bar{x})) = \frac{C}{(|x||y-\bar{x}|)^{n-2}}$$

Now consider the expressions,

$$\begin{aligned} (|x||y-\bar{x}|)^2 &= |x|^2|y-\bar{x}|^2 \\ &= |x|^2\left(|y|^2 - 2y\cdot\bar{x} + |\bar{x}|^2\right) \\ &= |x|^2\left(1 - 2y\cdot\left(\frac{x}{|x|^2}\right) + \frac{1}{|x|^2}\right) \\ &= |x|^2 - 2y\cdot x + 1 \\ &= |x|^2 - 2y\cdot x + |y|^2 \\ &= (|x-y|)^2, \end{aligned}$$

and so $(|x||y-\bar{x}|)^2 = (|x-y|)^2$. It then follows that,

$$\Phi_x = \Phi(|x|(y-\bar{x})) = \frac{C}{(|x||y-\bar{x}|)^{n-2}} = \frac{C}{|x-y|^{n-2}} = \Phi(y-x).$$

Therefore $\Phi_x = \Phi(y-x)$ on the boundary.

Using the definition of Green's function i.e. $G(x,y) = \Phi(y-x) - \Phi_x(y)$. We may then explicitly compute this. Symbolically, for our case we have,

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\bar{x})). \tag{2.18}$$

We may now express $u(x)$ i.e. the solution of Laplace's equation on the; in terms of (2.18). Using the representation formula (2.15),

$$u(x) = -\int_{\partial B} g(y)\frac{\partial G(x,y)}{\partial \hat{\mathbf{n}}}\, dS(y), \tag{2.19}$$

Recall that we are on the unit ball and so the normal vector at y is simply y. Hence,

$$u(x) = -\int_{\partial B} g(y) DG(x,y) \cdot y \, dS(y).$$

We must then compute $DG(x,y)$.

$$G_{y_i} = \Phi_{y_i}(y-x) - \left(\Phi^x(|x|(y-\bar{x}))\right)_{y_i}.$$

Let the first term on the right be A and the second term B. Recall that,

$$\Phi(x) = \frac{1}{n\alpha(n)(n-2)} \frac{1}{x^{n-2}},$$

then,

$$\phi_{y_i}(x) = \frac{1}{n(n-2)\alpha(n)}(2-n)|x|^{1-n}\frac{x_i}{|x|} = \frac{-x_i}{n\alpha(n)|x|^n}.$$

With this in mind, we may calculate A and B.

$$A = \Phi_{y_i}(y-x) = \frac{x_i - y_i}{n\alpha(n)|x-y|^n}.$$

And then B,

$$\left(\Phi^x(|x|(y-\bar{x}))\right)_{y_i} = \Phi_{y_i}(|x|(y-\bar{x}))|x|,$$

by the chain rule. Then using $\Phi_{y_i}(x)$,

$$\Phi_{y_i}(|x|(y-\bar{x}))|x| = \frac{-\left(|x|\left(y_i - \frac{x_i}{|x|^2}\right)\right)|x|}{n\alpha(n)||x|(y-\bar{x})|^2} = \frac{-(y_i|x|^2 - x_i)}{n\alpha(n)|x-y|^n}.$$

Ultimately, we may write down the derivative of our Green's function as,

$$\begin{aligned} G_{y_i}(x,y) &= A - B \\ &= \frac{1}{n\alpha(n)|x-y|^n}\left(x_i - y_i + y_i|x|^2 - x_i\right) \\ &= \frac{1}{n\alpha(n)|x-y|^n}\left(|x|^2 - 1\right) y_i. \end{aligned}$$

Now that we have the gradient of the Green's function, we need only evaluate the dot product with y to gain the normal derivative,

$$\begin{aligned} \frac{\partial G}{\partial \hat{\mathbf{n}}} &= DG \cdot y \\ &= \sum_{i=1}^n G_{y_i} y_i \\ &= \sum_{i=1}^n \left(\frac{\left(|x|^2-1\right) y_i}{n\alpha(n)|x-y|^n}\right) y_i \\ &= \frac{\left(|x|^2-1\right)}{n\alpha(n)|x-y|^n}\sum_{i=1}^n y_i^2 \\ &= \frac{\left(|x|^2-1\right)}{n\alpha(n)|x-y|^n}. \end{aligned}$$

Finally, the representation formula gives us,

$$\begin{aligned} u(x) &= -\int_{\partial B(0,1)} g(y)\frac{\left(|x|^2-1\right)}{n\alpha(n)|x-y|^n}\, dS(y) \\ &= \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, dS(y). \end{aligned} \tag{2.20}$$

We may then use a change of variable to solve Laplace on any $B(0,r)$. Upon doing so, we gain Poisson's formula for the ball,

$$u(x) = \frac{r^2 - |x|^2}{rn\alpha(n)} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, dS(y). \tag{2.21}$$

We call the term,

$$\frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n},$$

the *Poisson kernel* for the ball.

2.5.2 Energy Methods

The idea of energy methods for PDEs is to base all analysis on the governing PDE itself rather than use symmetries and then deduce what particular solutions may look like. In turn this allows us to deduce powerful results about behaviour of solutions rather than prove their existence.

For a more general outlook on the world of PDEs, it turns out that energy methods are very powerful and useful. This is since, to some extent, they can be used to derive results on all kinds of PDEs including higher order, non-linear ones. We now apply an energy method to prove uniqueness of Laplace's equation on some domain.

Theorem 2.12. ***Uniqueness Theorem:*** *Let $U \subset \mathbb{R}^n$ be an open, smooth set. The BVP,*

$$\begin{cases} -\Delta u & = f \;\; \text{in } U \\ u & = g \;\; \text{on } \partial U \end{cases}$$

has at most 1 solution.

Proof. Assume the BVP has two solutions u_1 and u_2 and let $w := u_1 - u_2$, we want to show that $w = 0$.

From the BVP it should be clear that w is harmonic and zero on the boundary. Then, applying the maximum principle $0 \leq w \leq 0$ implies $w = 0$. □

Let us now use an energy method to prove the claim. Given w as in the previous proof; multiplying w through the PDE and integrating over the domain, upon an integration by parts gives us,

$$\int_U -\Delta w = -\int_{\partial U} \frac{\partial w}{\partial \hat{\mathbf{n}}} w + \int_U |Dw|^2 \, dx = 0.$$

Note the boundary terms vanish since w must be homogenous on the boundary and since w is harmonic, equality to zero is justified. Notice, the integral over U on the left hand side is the L^2 norm of Dw and is therefore a non-zero positive map. Hence for the integral to equal zero it must be the case that $w = 0$. This constitutes an alternative proof of Theorem 2.12.

We now turn to another major application of energy methods concerning Laplace's equation.

Dirichlet's Principle

Here we want to show that the solution of Poisson's equation on U can be characterized as a minimizer of a certain functional[1]. We define the *energy functional*,

$$I[w] := \frac{1}{2} \int_U |Dw|^2 - wf \, dx. \tag{2.22}$$

[1]As it turn out this is an application of variational analysis i.e. the calculus of variations.

Here w belongs to an admissible class as defined by,

$$\mathcal{A} := \{w \in C^2(\bar{U}) : w = g \text{ on } \partial U\}.$$

In general, we can view Dirichlet's principle and by extension the calculus of variations as connecting minimizers of energy to solutions of PDEs. This idea is very deep and also has a beautiful connection to physics.

Theorem 2.13. ***Dirichlet Principle:*** *If $u \in C^2(\bar{U})$, then u solves Poisson's equation on U if and only if,*

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

[2]

Proof. Suppose u minimizes $I[u]$, we will show that u solves the Poisson BVP. Fix any smooth compactly supported function on U i.e. $v \in C_c^\infty(U)$. Define the functional,

$$I[u + \tau v] := \int_U \frac{1}{2}|Du + \tau Dv|^2 - f(u + \tau v)\, dx, \tag{2.23}$$

here $\tau \in \mathbb{R}$. Notice then that the equation above is a real-valued function of τ i.e. $i(\tau) :=$ (2.23). Furthermore, for every τ, $u + \tau v \in \mathcal{A}$. Also, we assumed at u $I[u]$ is minimized hence for every other $\tau \neq 0$, $i(\tau)$ is greater than $I[u]$. Since $i(\tau)$ is a single variable real function and has a minimum at $\tau = 0$ is must be the case that $i'(0) = 0$. Calculating this explicitly,

$$i'(\tau) = \int_U |Du + \tau Dv| \cdot Dv - fv\, dx.$$

Then we have that,

$$i'(0) = \int_U Du \cdot Dv - fv\, dx = 0.$$

Integrating the first term by parts and noting that $v = 0$ on the boundary we have that,

$$\int_U (-\Delta u)v - fv = 0.$$

For the equation to be true it must be the case that $(-\Delta u)v = fv \implies -\Delta u = f$. Recalling that $u \in \mathcal{A}$ ensures that boundary information is taken into account and so $u = g$ on ∂U.

For the reverse direction suppose that u solves the Poisson BVP, we want to show that $I[u]$ is the minimizer of $I[w]$. Explicitly, we want to show $I[u] \leq I[w]$. Consider $u - w$ and note that the function must be zero on the boundary. Testing this function against the PDE and integrating over the domain,

$$\int_U (-\Delta u - f)(u - w)\, dx = 0,$$

it follows that,

$$\int_U -\Delta u(u - w)\, dx = \int_U f(u - w)\, dx.$$

Integrating the left expression by parts and noting that $u - w$ vanishes on the boundary,

$$\int_U |Du|^2 - Du \cdot Dw\, dx = \int_U f(u - w)\, dx$$
$$\int_U |Du|^2 - fu\, dx = \int_U Du \cdot Dw - fw\, dx.$$

[2]As it happens, Poisson's equation can be shown to be the Euler-Lagrange equation for the minimization of the energy functional in Dirichlet's principle. In this specific case however, the converse is also true i.e. a solution of Poisson's equation solve the minimization problem.

Now, consider the following string of inequalities involving $Du \cdot Dv$,

$$\int_U Du \cdot Dv \, dx \le \int_U |Du \cdot Dw| \, dx \le \int_U |Du||Dw| \, dx \le \int_U \frac{1}{2}|Du| + \frac{1}{2}|Dw| \, dx.$$

The first inequality simply comes from the fact that any scalar function is at most as large as its absolute value. The second inequality is an application of Cauchy-Schwarz inequality and the final inequality comes from Cauchy's inequality[3].

Returning to where we left off, it must be the case that,

$$\int_U |Du|^2 - fu \, dx \le \int_U \frac{1}{2}|Du| + \frac{1}{2}|Dw| - fw \, dx$$
$$\int_U \frac{1}{2}|Du|^2 - fu \, dx \le \int_U \frac{1}{2}|Dw| - fw \, dx,$$

Hence, $I[u] \le I[w]$, which establishes the claim. □

2.6 Heat equation

We now study the heat equation,

$$u_t - \Delta u = 0. \tag{2.24}$$

The non-homogeneous version of the equation is of course given by f replacing zero.

By analogy to Laplace's equation, we follow the guiding principle that any statement of the fundamental solution of Laplace's equation yields a similar but more complicated statement about the heat equation. Recall, for Laplace's equation, solutions are radial. In a similar manner we deduce that solution may involve the term $\frac{|x|}{\sqrt{t}} = 1$.

2.6.1 Derivation of fundamental solution

Let $u = \frac{1}{t^{n/2}} v(\frac{x}{\sqrt{t}})$ and assume that v is radial. Then we consider u to be our ansatz.

Substituting into the PDE,

$$\left(t^{-n/2} v\left(\frac{x}{\sqrt{t}}\right)\right)_t = \Delta\left(t^{-n/2} v\left(\frac{x}{\sqrt{t}}\right)\right)$$
$$\frac{-n}{2} t^{-n/2-1} v\left(\frac{x}{\sqrt{t}}\right) + t^{-n/2} v\left(\frac{x}{\sqrt{t}}\right)_t = t^{-n/2}\frac{1}{t}\Delta v\left(\frac{x}{\sqrt{t}}\right).$$

Now, let us take special care in evaluating the time derivative.

$$\begin{aligned} v\left(\frac{x}{\sqrt{t}}\right)_t &= v\left(\frac{x_1}{\sqrt{t}} \dots \frac{x_n}{\sqrt{t}}\right)_t \\ &= v_{x_1}\left(\frac{-x_1}{2t\sqrt{t}}\right) + \dots + v_{x_n}\left(\frac{-x_n}{2t\sqrt{t}}\right) \\ &= \frac{-1}{2t}\left(v_{x_1}\left(\frac{x_1}{\sqrt{t}}\right) + \dots + v_{x_n}\left(\frac{x_n}{\sqrt{t}}\right)\right) \\ &= \frac{-1}{2t} Dv \cdot \left(\frac{x}{\sqrt{t}}\right). \end{aligned}$$

[3] Given $a, b \in \mathbb{R}$, $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$.

Let $y \equiv \frac{x}{\sqrt{t}}$. Then, we can write out the entire equation as,

$$\begin{aligned} -\frac{n}{2}t^{-n/2-1}v\left(y\right)-\frac{t^{-n/2}}{2t}Dv\cdot y&=t^{-n/2}\frac{1}{t}\Delta v(y)\\ -\frac{n}{2}t^{-n/2-1}v-\frac{t^{-n/2-1}}{2}Dv\cdot y&=t^{-n/2-1}\Delta v\\ \frac{n}{2}v-Dv\cdot y+\Delta v.&=0. \end{aligned} \tag{2.25}$$

We now have a PDE in one less variable i.e. there are no terms with t or the t derivative.

We will now make this single variable PDE into an ODE. We use the fact that v is assumed radial i.e. $v(r) = w(r)$ where $w : \mathbb{R}^+ \to \mathbb{R}$. Then,

$$Dv = w'(r)\frac{y}{|y|},$$

and

$$\Delta v = w'' + \left(\frac{n-1}{r}\right)w'.$$

Hence, re-writing (2.25) in terms of w,

$$\frac{n}{2}w + \frac{1}{2}w'\frac{y}{|y|}\cdot y + w'' + \left(\frac{n-1}{r}\right)w' = 0.$$

Simplifying we get,

$$\frac{1}{2}\left(w'r + wn\right) + w'' + \left(\frac{n-1}{r}\right)w' = 0. \tag{2.26}$$

Now we need only solve the ODE (2.26). To do so however, we need the realization that (2.26) resembles the product rule for differentiation. Recalling some techniques from ODEs, let us multiply by the integrating factor xr^{n-1},

$$\begin{aligned} \frac{1}{2}\left(w'r^n + wnr^{n-1}\right) + w''r^{n-1} + w'(n-1)r^{n-2} &= 0\\ \frac{1}{2}(r^n w)' + (w'r^{n-1})' &= 0\\ \frac{d}{dr}\left(\frac{1}{2}r^n w + w'r^{n-1}\right) &= 0 \end{aligned}$$

Therefore, it must be the case that,

$$\frac{1}{2}r^n w + w'r^{n-1} = C,$$

where C is a constant. But since we seek a particular solution, let us simplify things and take $C = 0$. We then have a nice separable ODE in w with solution,

$$w = \exp\left(-\frac{r^2}{4} + C\right) = C\exp\left(-\frac{r^2}{4}\right).$$

Now let us go back to v and finally u variables.

$$v(y) = w(|y|) = C\exp\left(-\frac{|y|^2}{4}\right),$$

finally, going back to u,

$$u(x,t) = \frac{1}{t^{n/2}}v\left(\frac{x}{\sqrt{t}}\right) = \frac{C}{t^{n/2}}\exp\left(-\frac{|x|^2}{4t}\right).$$

To give u the property of integral unity over $\mathbb{R}^n$, we set $C = 4\pi^{-n/2}$.

Definition 2.14. *The function,*

$$\Phi(x,t) := \begin{cases} \frac{1}{4\pi t^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & (x \in \mathbb{R}^n,\ t > 0) \\ 0 & (x \in \mathbb{R}^n,\ t < 0) \end{cases} \tag{2.27}$$

is the fundamental solution for the heat equation.

We also call (2.27) the *heat kernel.*

2.6.2 Initial Value Problem

We now turn to solving the heat equation within the context of an intial value problem. Much like the solution of Laplace's equation on some bounded domain, proving a solution to the IVP will require careful analysis. Furthermore, it will be built upon the fundamental solution of the heat equation.

We wish to solve the problem,

$$\begin{cases} u_t - \Delta u = 0 \text{ in} & \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \text{ in} & \mathbb{R}^n \times (t=0) \end{cases} \tag{2.28}$$

Recall the procedure of solving Poisson's equation; we took the fundamental solution and convolved with the inhomogeneous term[4]. Here we do the same i.e. take the fundamental solution of the heat equation and convolve with the inhomogeneous term i.e. $g(x)$ and show that the result satisfies (2.28).

$$u(x,t) = \Phi * g(x) = \int_{\mathbb{R}^n} \Phi(y-x)g(y)\,dy. \tag{2.29}$$

Theorem 2.15. *Let $g \in C(\mathbb{R}^n) \cap L^\infty$ and let u be as in* (2.29)*. Then,*

- $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$.
- $u_t(x,t) - \Delta u(x,t) = 0$ $x \in \mathbb{R}^n$ *and* $t > 0$.
- $\lim_{(x,t)\to(x_0,0)} = g(x)$ *for each* $x_0 \in \mathbb{R}^n$.

Proof. Let us begin by first using the heat kernel,

$$K(y) = \frac{1}{(4\pi)^{n/2}} \exp\left(-\frac{|y|^2}{4}\right).$$

Observe first that K belongs to Schwarz class[5]. K also has the property that its mass is unity. With these in mind, we may re-write u in term of K as,

$$u(x,t) = \frac{1}{t^{n/2}} K * g.$$

Using K as a standard mollifier it follows that $u \in C^\infty$ proving the fist claim.

Now let us show that $u_t = \Delta u$. Consider the Laplacian term, we already know, that the spatial derivatives commute with the integral and hence,

$$\Delta u(x,t) = \Delta\left(\frac{1}{t^{n/2}} K * g\right) = \frac{1}{t^{n/2}} \Delta K * g.$$

[4]This was heuristically justified as being analogous to the superposition principle.
[5]Schwarz class are smooth functions that decay faster than any polynomial and ∞.

Consider then the time derivative. We want to show that the time derivative commutes with the integral i.e. $\partial_t \int_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \partial_t$. In order to do this we must employ the dominated convergence theorem. Let's fix $t_0 > 0$ and consider the string of inequalities,

$$\left(1 + |x-y|^{n+2}\right) t^{-n/2-1} \exp\left(\frac{-|x-y|^2}{4t}\right) \leq t_0^{-n/2-1} + \left(\frac{|x-y|^2}{t}\right)^{n/2+1} \exp\left(\frac{-|x-y|^2}{4t}\right) \leq t_0^{-n/2-1} + C_n.$$

The last inequality follows from the fact that K belongs to the Schwarz class and is hence bounded by some constant depending on n. Clearly the dominating function is integrable and hence by the dominated convergence theorem, we may pass the limit under the integral. Therefore the time derivative commutes with the integral.

Therefore we have that,

$$u_t(x,t) = \partial_t \left(\frac{1}{t^{n/2}} K * g\right) = \Phi_t * g.$$

Hence, $u_t = \Delta u$ and the second claim is established.

We now want to show that we recover the initial condition as we drive $t \to 0$. Our goal is to show that if $|x - x_0|$ and $|t|$ is small enough, then $|u(x,t) - u(x_0, 0)|$ is small enough. This should hint at using an epsilon-delta type argument.

Fix $x_0 \in \mathbb{R}^n$ and let $\epsilon > 0$.By continuity, there exists $\delta > 0$ such that if $|y - x_0| < \delta$ then $|g(y) - g(x_0)| < \epsilon$. Then, if $|x - x_0| < \delta/2$ then,

$$|u(x,t) - g(x_0)| = \left|\int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy - \int_{\mathbb{R}^n} \Phi(x-y,t) g(x_0)\, dy\right| = \left|\int_{\mathbb{R}^n} \Phi(x-y,t)\left(g(y) - g(x_0)\right)\, dy\right|.$$

The second inequality follows from the fact that $g(x_0) = g(x_0)1 = g(x_0)\int_{\mathbb{R}^n} \Phi(x-y) dy$, since the fundamental solution has mass unity. So,

$$\left|\int_{\mathbb{R}^n} \Phi(x-y,t)\left(g(y) - g(x_0)\right)\, dy\right| \leq \int_{\mathbb{R}^n} \Phi(x-y,t)|g(y) - g(x_0)|\, dy.$$

Notice that if y is close to x_0, then $|g(y) - g(x_0)|$ becomes small. However there is no way of knowing wether the points are close or far. Hence, we must decompose the integral into two regions.

$$\int_{\mathbb{R}^n} \Phi(x-y,t)|g(y) - g(x_0)|\, dy = \left(\int_{B(x_0,\delta)} + \int_{\mathbb{R}^n \setminus B(x_0,\delta)}\right) \Phi(x-y,t)|g(y) - g(x_0)|\, dy.$$

Consider first the integral over the δ-ball. Here, the points are close together and hence $|g(y) - g(x_0)| \leq \epsilon$, so

$$\int_{B(x_0,\delta)} \Phi(x-y,t)|g(y) - g(x_0)|\, dy \leq \epsilon \int_{B(x_0,\delta)} \Phi(x-y,t)\, dy.$$

Using the fact that Φ has mass unity over the whole space provides a strict inequality,

$$\epsilon \int_{B(x_0,\delta)} \Phi(x-y,t)\, dy < \epsilon.$$

Now consider the integral over $\mathbb{R}^n \setminus B(x_0,\delta)$. Here we want to exploit the fact that Φ is a Schwarz class function and hence decays to zero in the far field. Moreover, recall that $g \in L^\infty$, hence is bounded above. So,

$$\int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)|g(y) - g(x_0)|\, dy \leq C \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(y-x,t)\, dy$$

Recall that $|x-x_0| < \frac{\delta}{2}$ and if $|y-x_0| > \delta$ then it must be that $|y-x_0| < 2|y-x|$. To prove this geometric fact, we may use the triangle inequality,

$$|y-x_0| \leq |y-x| + \underbrace{|x-x_0|}_{<\delta/2} < |y-x| + \frac{|y-x_0|}{2}.$$

Solving for $|y-x_0|$ we get,

$$\left(1-\frac{1}{2}\right)|y-x_0| < |y-x| \implies \frac{1}{2}|y-x_0| < |y-x| \implies |y-x_0| < 2|y-x|.$$

We may then exploit this fact.

$$C\int_{\mathbb{R}^n\setminus B(x_0,\delta)} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|y-x|^2}{4t}\right)\, dy < C\int_{\mathbb{R}^n\setminus B(x_0,\delta)} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|y-x_0|^2}{16t}\right)\, dy.$$

Notice we have used our derived geometric fact regarding distances between y, x and x_0.

Now let's change variables. Write the exponential as $\exp\left(\frac{|y-x_0|}{\sqrt{16t}}\right)^2$. Let $z = \frac{|y-x_0|}{\sqrt{16t}}$ and so $dz = \frac{dy}{(16t)^{n/2}} = \frac{dy}{2n(4t)^{n/2}}$. Hence the integral of the far field can be written in the new variable as,

$$\int_{\mathbb{R}^n\setminus B\left(0,\frac{\delta}{\sqrt{16t}}\right)} \frac{2^n}{\pi^{n/2}} \exp\left(-z^2\right)\, dz.$$

Notice now that as $t \to 0$, $\frac{\delta}{\sqrt{16t}} \to \infty$ i.e. the radius of the ball gets larger. Then, since the integral over the entire space is finite, it must be the case that as the ball gets larger the entire integral goes to zero.

Hence, $|u(x,t) - g(x_0)| < 2\epsilon$ and we are done. $\square$

2.6.3 Non-homogenous Problem

We now turn our attention to solving the non-homogeneous IVP for the heat equation.

$$\begin{cases} u_t - \Delta u = f \text{ in } & \mathbb{R}^n \times (0,\infty) \\ u = 0 \text{ on } & \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{2.30}$$

We take the motivation, that for some fixed s,

$$u = u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy,$$

solve the homogenous IVP,

$$\begin{cases} u_t - \Delta u = 0 \text{ in } & \mathbb{R}^n \times (0,\infty) \\ u = f(\cdot, s) \text{ on } & \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{2.31}$$

Notice that (2.31) is of the form (2.28). And so using a generalized super-position principle, we may be able to construct solutions of (2.30) through the solution of (2.31). This idea is known as *Duhamel's principle.*

Precisely, consider the solution,

$$u(x,t) = \int_0^t u(x,t;s)\, ds \quad (x \in \mathbb{R}^n,\ t \geq 0).$$

But then, taking the assertion of the form of $u(x,t;s)$,

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy\, ds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) f(y,s)\, dy\, ds. \end{aligned} \tag{2.32}$$

Now we would like to prove that (2.32) indeed is a solution to (2.30).

Theorem 2.16. ***Solution to Non-Homogenous Heat IVP:*** *Let $f \in C_c^2(\mathbb{R}^n \times [0,\infty))$ and let u be as in* (2.32)*, then;*

- $u \in C_c^2(\mathbb{R}^n \times [0,\infty))$.
- $u_t - \Delta u = f$*, for $t > 0$ and $x \in \mathbb{R}^n$.*
- $\lim_{(x,t)\to(x_0,0)} = 0$ *for each $x_0 \in \mathbb{R}^n$.*

Proof. Recall that Φ has a singularity at the origin, so we may not naively differentiate under the integral.

We introduce the change of variables,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\, dy\, ds.$$

Recall that f has compact support and Φ is smooth near $t = s$. Hence,

$$u_t = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy.$$

Similarly for the spatial derivatives,

$$u_{x_i x_j} = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_i x_j}(x-y,t-s)\, dy\, ds.$$

Hence the first claim is established.

Now we want to show that $u_t - \Delta u = f$. Keeping in mind the new variables,

$$\begin{aligned} u_t - \Delta u &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \left[(\partial_t - \Delta) f(x-y,t-s)\right]\, dyds + \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,0)\, dyds \\ &= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \left[(-\partial_s - \Delta) f(x-y,t-s)\right]\, dyds + \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s) \left[(-\partial_s - \Delta) f(x-y,t-s)\right]\, dyds \\ &+ \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,0)\, dyds \end{aligned}$$

Now let I_ϵ, J_ϵ and K be the functionals on the last line above respectively.

Consider first the J_ϵ functional. Since f has compact support it must be bounded and hence there exists some constant C[6] such that,

$$|J_\epsilon| \le C \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(y,s)\, dyds = \epsilon C.$$

The equality follows by recalling the fact that the fundamental solution have integral unity. Hence taking $\epsilon \to 0$ implies $J_\epsilon \to 0$.

[6] This constant can be given for example by sup norm of f.

Consider then I_ϵ. Following an integration by parts, it is easy to see that,

$$\begin{aligned} I_\epsilon &= \int_\epsilon^t \int_{\mathbb{R}^n} \left[(\partial_s - \Delta)\, \Phi(y,s) \right] f(x-y, t-s)\, dyds + \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\, dy - K \\ &= \int_{\mathbb{R}^n} \Phi(y,\epsilon) f(x-y,t-\epsilon)\, dy - K. \end{aligned}$$

The second line follows as the integral over space and time must be zero since Φ solves the heat equation. Now taking $\epsilon \to 0$, the integrand of the space integral in the last line becomes $\Phi(y,0)$. But this is simply the Dirac delta distribution and hence the integral over $\mathbb{R}^n$ evaluates to f. So, we have that as $\epsilon \to 0$,

$$u_t - \Delta u = f + K - K = f.$$

This establishes the second claim.

Lastly, we want to show that,

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = 0,$$

for each $x_0 \in \mathbb{R}^n$. We have that $\|u(\cdot,t)\|_{L^\infty} \leq t\|f\|_{L^\infty} \to 0$ as $t \to 0$. Which establishes all claims. □

For the most general IVP of the form,

$$\begin{cases} u_t - \Delta u = f \text{ in } & \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g \text{ in } & \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{2.33}$$

we may use the last two theorems to deduce a representation formula of the form,

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dyds \tag{2.34}$$

2.6.4 Mean Value Properties of the Heat Equation

We formulate the **maximum principle** for the heat equation. Physically, the principle states, that given some closed, bounded domain; if we heat the boundary, then the hottest the temperature of the domain can be is the temperature of the boundary.

We may formulate the minimum principle analogously.

Theorem 2.17. ***Strong Maximum Principle:*** *Assume $u \in C_1^2$ and solves the heat equation within the cylinder U_T, then*

- *Then $\max_{U_T} u = \max_{\Gamma_T} u$.*
- *Also, if U is connected and there exists a point $(x_0,t_0) \in U_T$ such that,*

$$u(x_0,t_0) = \max_{\bar{U}_T} u,$$

then u is constant on U_T.

2.6.5 Energy Methods

We now look into some energy methods for the heat equation. We may use an energy method to show uniqueness of solutions of the heat equation.

Theorem 2.18. ***Uniqueness of solutions of the heat equation:*** *There exists only one solution $u \in C_1^2$ of the problem,*

$$\begin{cases} u_t - \Delta u & = f \text{ in } U_T \\ u & = g \text{ on } \Gamma_T. \end{cases} \tag{2.35}$$

Proof. Suppose there exists another solution $\tilde{u}$ and define $w := u - \tilde{u}$. Then (2.35) transforms as,

$$\begin{cases} w_t - \Delta w & = 0 \text{ in } U_T \\ w & = 0 \text{ on } \Gamma_T. \end{cases}$$

We want to show $w = 0$ solves (2.35). We define the energy integral over space in the usual way,

$$E(t) = \int_U w^2(x,t)\,dx, \quad 0 \le t \le T.$$

An application of the chain-rule gives the time derivative; then using the PDE and integrating by parts

$$\frac{d}{dt}E(t) = 2\int_U w \cdot \underbrace{w_t}_{=\Delta w,\text{ by PDE}}\,dx = 2\int_U w\Delta w\,dx = -2\int_U |Dw|^2\,dx.$$

Here we have assumed enough regularity to interchange the derivative and integral since $u \in C_1^2$.

Since the integral of the final result is sign-definite, it must be the case that $\frac{d}{dt}E(t) \le 0$. Hence $E(t) \le E(0) = 0$ and so $w = 0$ proving the claim. □

We can also ask about uniqueness of solution propagating backwards in time. Suppose that u and $\tilde{u}$ are smooth solutions to the IVBVPs,

$$\begin{cases} u_t - \Delta u & = f \text{ in } U_T \\ u & = g \text{ on } \partial U \times [0,T]. \end{cases} \tag{2.36}$$

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} & = f \text{ in } U_T \\ \tilde{u} & = g \text{ on } \partial U \times [0,T]. \end{cases} \tag{2.37}$$

Note that we do not assume that $u = \tilde{u}$ at $t = 0$ i.e. the smooth solutions may have different initial conditions. However, amazing this will turn out to be true. Physically, this means that is two temperature distributions agree as some $T > 0$ point in time on the domain U and have the same boundary values for all $0 \le t \le T$, then it must be the case that the distributions agree at all times on the domain!

Theorem 2.19. ***Backwards Uniqueness:*** *Suppose $u, \tilde{u} \in C^2$ solves the IBVPs above. If $u(x,T) = \tilde{u}(x,T)$ then $u \equiv \tilde{u}$ in U_T.*

Proof. Define $w := u - \tilde{u}$ and also define the energy functional $E(t)$ in the usual ways as in the previous proof. We know that $\dot{E}(t) \le 0$. Taking second derivatives, we have,

$$\begin{aligned} \ddot{E}(t) &= -4\int_U Dw \cdot Dw_t\,dx \\ &= 4\int_u \Delta w w_t\,dx \\ &= 4\int_u (\Delta w)^2\,dx. \end{aligned}$$

The first line uses regularity to interchange derivatives and integrals and then uses the chain-rule. The second line follows via an integration by parts and knowing w vanishes on the boundary as dictated by the IBVP in w. The last line follows from the fact that $\Delta w = w_t$.

From the previous integration by parts we also know,

$$-\int_U |Dw|^2\,dx = \int_U w\Delta w\,dx \le \left(\int_U w^2\,dx\right)^{1/2}\left(\int_U (\Delta w)^2\,dx\right)^{1/2},$$

where the inequality follows via an application of Holder's inequality. Then we have that,

$$(\dot{E}(t))^2 = 4\left(\int_U |Dw|^2\,dx\right)^2 \leq \left(\int_U w^2\,dx\right)\left(4\int_U (\Delta w)^2\,dx\right) = E(t)\ddot{E}(t).$$

Clearly, $\dot{E}^2 \leq E\ddot{E}$. Now if $E = 0$ for all t, then we are done.

Otherwise there must exist $[t_1, t_2] \subset [0, T]$ with $E(t) > 0$ for $t_1 \leq t < t_2$ and $E(t_2) = 0$.

If we then define $f(t) := \log E(t)$ for $t_1 \leq t < t_2$ then,

$$\ddot{f} = \frac{\ddot{E}}{E} - \frac{\dot{E}^2}{E^2} \geq 0.$$

Hence it must be that f is convex over the interval (t_1, t_2). □

2.7 Wave equation

We now turn to looking at the final of the four important linear PDEs in two variables. The wave equation takes the form,

$$u_{tt} - \Delta u = 0. \tag{2.38}$$

Where the non-homogenous counterpart of the equation is given when the right-had side is equal to some function $f : \mathbb{R}^n \to \mathbb{R}$. We often make use of the abbreviation,

$$\Box u := u_{tt} - \Delta u,$$

where the operator $\Box$ is called the d'Alemberitian.

2.8 A Physical Derivation

2.9 d'Alembert's Solution for $n = 1$

We consider here the IVP involving the wave equation on $\mathbb{R}$,

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in} & \mathbb{R} \times (0, \infty) \\ u = g,\ u_t = h \text{ on} & \mathbb{R} \times \{t = 0\}. \end{cases} \tag{2.39}$$

Notice, due to the second time derivative, we need to initial conditions, on u and u_t for the IVP to be well-posed.

The key observation here is that,

$$\Box = \partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x).$$

We may then write the wave equation as,

$$(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0,$$

if we then define,

$$v(x, r) := (\partial_t - \partial_x)u(x, t),$$

The wave equation itself transforms in terms of v to,

$$(\partial_t + \partial_x)v(x, t) = 0.$$

Notice what has just occurred. We transformed the wave-equation into the linear transport equation. Hence the price to solve the wave equation in $n = 1$ is the same as solving two transport equations, the first is non-homogeneous and the second is homogeneous in terms of v.

The solution to the homogenous transport equation is simply $v(x,t) = a(x-t)$ in $\mathbb{R} \times (0,\infty)$. Then the non-homogenous transport equation in terms of u can be written as,

$$u_t - u_x = a(x-t).$$

We may solve for u taking into account the initial data; by using the representation solution for the wave equation.

$$u(x,t) = g(x+t) + \int_0^t a(x-(s-t)-s)\,ds = g(x+t) + \int_0^t a(x-2s+t)\,ds.$$

This is quite ugly, let's change variables. Let $s' = x - 2s + t$ then $ds' = -s\,ds$ and so $ds = -\frac{1}{2}ds'$. Moreover, $s'(0) = x+t$ and $s'(t) = x+t-2t = x-t$. Hence,

$$u(x,t) = g(x+t) - \frac{1}{2}\int_{x+t}^{x-t} a(s')\,ds' = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(s')\,ds'.$$

Now we must figure out what the arbitrary function a is using the initial data. It is clear to see that $u(x,0) = g(x)$, since the integral will vanish. But this does not tell us anything about a. We will need the second initial condition.

Recall $u_t(x,0) = h(x)$. Differentiating the representation solution,

$$u_t(x,t) = g'(x,t) + \frac{1}{2}\left(a(x+t) + a(x-t)\right).$$

Setting $t = 0$,

$$u_t(x,0) = h(x) = g'(x,0) + \frac{1}{2}\left(a(x) + a(x)\right) = g'(x,0) + a(x).$$

We also have that $a(x) = h(x) - g'(x)$. Then we may re-write u in terms of the intial data,

$$\begin{aligned} u(x,t) &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds - \frac{1}{2}(g(x+t) - g(x-t)) \\ &= \frac{1}{2}g(x+t) + \frac{1}{2}g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds. \end{aligned}$$

The last line is known as *d'Alembert's formula.*

2.9.1 Reflection of waves

We may apply d'Alembert's formula and consider the IBVP on the half-space $\mathbb{R}_+$.

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in} & \mathbb{R} \times (0,\infty) \\ u = g,\ u_t = h \text{ on} & \mathbb{R} \times \{t=0\} \\ u = 0 \text{ on} & \{x=0\} \times \{t=0\} \end{cases} \tag{2.40}$$

Physically, this IBVP describes a one dimensional rod such that there is a rigid boundary on one end i.e. $x = 0$ and an open/free boundary on the other $x \to \infty$.

We are given the initial position and initial velocity of this wave by g and h respectively. Furthermore, we known part of the travelling wave will hit the wall at $x = 0$. Since there is no dissipation in the system, this wave must be reflected back and travel towards ∞. The challenge is then to show this behaviour mathematically.

Since we need the wave to vanish at the point $x = 0$ we will take the odd-reflection of both g and h.

$$g_{\text{odd}} = \begin{cases} g(x) & x > 0 \\ 0 & x = 0 \\ -g(-x) & x < 0. \end{cases}$$

$$h_{\text{odd}} = \begin{cases} h(x) & x > 0 \\ 0 & x = 0 \\ -h(-x) & x < 0. \end{cases}$$

With this reflection in mind, we want to solve (2.40) on $\mathbb{R}$. To do this we make use of d'Alembert's formula. So,

$$u(x,t) = \frac{1}{2}\left(g_{\text{odd}}(x-ct) + g_{\text{odd}}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h_{\text{odd}}(s)\, ds.$$

It should be noted that clearly this solves the problem on the half-space as it solves it on the whole space.

Notice, that the argument of g i.e. $x - ct$ will switch sign. To take this into account, we split to cases.

Case 1: $x - ct > 0$, which implies $t < \frac{x}{c}$ or t is small. Hence,

$$u(x,t) = \frac{1}{2}\left(g(x-ct) + g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\, ds,$$

which is the usual d'Alembert's formula. This makes physical sense as for small times, the wave has not had enough time to hit the wall at $x = 0$ and reflect back. Hence the only solution to the wave equation on the line should be the one we know.

Case 2: $x - ct < 0$, which implies $t > \frac{x}{c}$ or t is large. Now we have that,

$$u(x,t) = \frac{1}{2}\left(g_{\text{odd}}\underbrace{(x-ct)}_{<0} + g_{\text{odd}}(x+ct)\right) + \frac{1}{2c}\int_{\underbrace{x-ct}_{<0}}^{x+ct} h_{\text{odd}}(s)\, ds.$$

And so we may change the formula as follows,

$$u(x,t) = \frac{1}{2}\left(-g(-x+ct) + g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{0} -h(-s)\, ds + \frac{1}{2c}\int_{0}^{x+ct} h(s)\, ds.$$

If we take $z = -s$ we get that,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(-g(-x+ct) + g(x+ct)\right) + \frac{1}{2c}\int_{-x+ct}^{0} h(z)\, ds + \frac{1}{2c}\int_{0}^{x+ct} h(s)\, ds \\ &= \frac{1}{2}\left(-g(-x+ct) + g(x+ct)\right) + \frac{1}{2c}\int_{-x+ct}^{x+ct} h(s)\, ds \end{aligned}$$

Here we see the $-x + ct = ct - x$ shows the wave after hitting the wall and being reflected towards $+\infty$.

2.9.2 Spherical means of solutions

Let $n \geq 2$, $m \geq 2$ and $u \in C^m(\mathbb{R}^m \times [0, \infty)$ solves the IVP,

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in} & \mathbb{R} \times (0, \infty) \\ u = g,\ u_t = h \text{ on} & \mathbb{R} \times \{t = 0\}. \end{cases} \tag{2.41}$$

Our aim is to derive an explicit representation solution for u in terms of g and h. The plan will be to study the average of u over certain spheres, where the averages are taken as functions of time and the radius. These in itself solve a PDE called the *Euler-Poisson-Darboux equation*. It turns out, for odd n, the EPD PDE transforms into the 1-D wave equation. Then applying the variant of d'Alembert's formula leads us to the solution.

In a larger picture what this tells us is that; in similar fashion to Laplace's equation we want to have some sort of mean value property for solutions to the wave equation. As it will turn out this is not the case directly speaking. However the EPD equation is sort of a mean value PDE for solutions of the wave equation as given by the IVP (2.41).

We denote the averages of u, g, h over some sphere $\partial B(x,r)$ in the usual way,

$$\begin{aligned} U(x;r,t) &:= \fint_{\partial B} u(y,t)\, dS(y) \\ G(x;r,t) &:= \fint_{\partial B} g(y,t)\, dS(y) \\ H(x;r,t) &:= \fint_{\partial B} h(y,t)\, dS(y) \end{aligned}$$

Lemma 2.20. ***Euler-Poisson-Darboux Equation:*** *Fix $x \in \mathbb{R}^n$ and let u solve* (2.41). *Then $U \in C^m(\mathbb{R}_+ \times [0,\infty)$ and*

$$\begin{cases} U_{tt} - Urr - \frac{n-1}{r}U_r = 0 \text{ in } & \mathbb{R} \times (0,\infty) \\ U = G,\ U_t = H \text{ on } & \mathbb{R}_+ \times \{t=0\}. \end{cases}$$

Note that $\partial_r^2 - \frac{n-1}{r}\partial_r$ is the radial part of the Laplacian in polar coordinates.

Proof. We begin by calculating a few partial derivatives on U,

$$U_r = \partial_r \left(\frac{1}{N\alpha(N) r^{N-1}} \int_{\partial B} u(y,t)\, dS(y) \right).$$

As we have seen before, it is best to change variables. Let $z = \frac{y-x}{r}$ and $dS(z) = \frac{1}{r^{N-1}} dS(y)$ so then $S(y) = r^{N-1} dS(z)$. Hence we may re-write,

$$\begin{aligned} U_r &= \partial_r \left(\frac{1}{N\alpha(N)} \int_{\partial B(0,1)} u(x+rz,t)\, dS(z) \right) \\ &= \frac{1}{N\alpha(N)} \int_{\partial B(0,1)} Du(x+rz,t) \cdot z\, dS(z) \\ &= \frac{1}{N\alpha(N)} \int_{\partial B(x,r)} Du(y,t) \cdot \left(\frac{x-y}{r} \right) \frac{1}{r^{N-1}} dS(y) \\ &= \frac{1}{N\alpha(N) r^{N-1}} \int_{\partial B(x,r)} Du(y,t) \cdot \hat{\mathbf{n}}\, dS(y) \\ &= \frac{r}{N} \fint_{B(x,r)} \Delta u(y,t)\, dy. \end{aligned}$$

Where the last line follows via integration by parts. Also, since u solves the wave equation, we deduce the relation,

$$U_r = \frac{r}{N} \fint_{B(x,r)} \Delta u(y,t)\, dy = \frac{r}{N} \fint_{B(x,r)} u_{tt}(y,t)\, dy.$$

Now we want to take a second derivative in r, but we have this pesky singularity for $r = 0$ coming from division by the surface measure. This however can be avoided by dividing both sides by r^{N-1}. So we have that,

$$U_r = \frac{1}{N\alpha(N) r^{N-1}} \int_{B(x,r)} u_{tt}(y,t)\, dy \implies r^{N-1} U_r = \frac{1}{N\alpha(N)} \int_{B(x,r)} u_{tt}(y,t)\, dy.$$

Now taking the derivative,

$$\partial_r \left(r^{N-1} U_r \right) = \frac{1}{N\alpha(N)} \int_{\partial B(x,r)} u_{tt}\, dy.$$

Now let us turn to taking time derivatives, which is much simpler.

$$U_{tt} = \frac{1}{N\alpha(N)r^{N-1}} \int_{\partial B(x,r)} u_{tt}(y,t)\, dy.$$

But notice, this is very similar to the expression $\partial_r \left(r^{N-1}U_r\right)$, we need only multiply by $\frac{1}{r^{N-1}}$. So,

$$U_{tt} = \frac{1}{r^{N-1}}\partial_r \left(r^{N-1}U_r\right) = \frac{1}{N\alpha(N)r^{N-1}} \int_{\partial B(x,r)} u_{tt}(y,t)\, dy.$$

Taking the first two expressions around the first equality and using the product rule to evaluate the derivative,

$$\begin{aligned}
U_{tt} &= \frac{1}{r^{N-1}}\partial_r \left(r^{N-1}U_r\right) \\
U_{tt} &= \frac{1}{r^{N-1}} \left((N-1)r^{N-2}U_r + r^{N-1}U_{rr}\right) \\
U_{tt} &= \frac{(N-1)}{r}U_r + U_{rr},
\end{aligned}$$

which is exactly the EPD PDE. □

The idea now is to solve for the $n = 3$ and then the $n = 2$ case for solving the wave equation. To do this, our strategy will be to transform the EPD equation into the wave equation for each of these dimensions.

2.9.3 Solution for $n = 3$ and Kirchhoff's Formula

We want to solve,

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in} & \mathbb{R}^3 \times (0,\infty) \\ u = g,\ u_t = h \text{ on} & \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{2.42}$$

We will solve this using EPD. Let $U(r,t)$ be defined as above, then from the previous section we know that U is a solution to the EPD PDE. Let,

$$\begin{aligned}
\tilde{U}(r,t) &= rU(r,t) \\
\tilde{G}(r) &= rG(r) \\
\tilde{H}(r) &= rH(r).
\end{aligned}$$

As it happens, $\tilde{U}$ solves,

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \text{ in} & (0,\infty) \times (0,\infty) \\ \tilde{U}(r,0) = \tilde{G},\ \tilde{U}_t(r,0) = \tilde{H} \text{ on} & \mathbb{R}^3 \times \{t = 0\} \\ \tilde{U}(0,t) = 0 \text{ on} & \{r = 0\} \times (0,\infty). \end{cases}$$

Notice, this problem is the same as solving the wave-equation on the positive real line, which we know how to solve via the method of reflections.

Let us show that $\tilde{U}$ does indeed solve this IBVP. Starting with the initial conditions,

$$\begin{aligned}
\tilde{U}(r,0) &= rU(r,0) = rG = \tilde{G}, \\
\tilde{U}_t(r,0) &= rU_t(r,0) = rH = \tilde{H}.
\end{aligned}$$

To show we recover the PDE, let us compute derivatives,

$$\tilde{U}_{tt} = rU_{tt} = r\left[U_{rr} + \frac{2}{r}U_r\right] = rU_{rr} + 2U_r = (U + rU_r)_r = \tilde{U}_{rr}.$$

It is important to note that the only reason we were able to identify the product rule is since we are in $n = 3$ dimensions. For a general n this realization would not be possible.

Finally lets check the boundary condition.

$$\tilde{U}(0,t) = \lim_{r \to 0^+} rU(r,t) = r \fint_{\partial B(x,r)} u(y,t)\, dS(y).$$

Now, since u is bounded, it follows that the average integral must also be bounded. Hence $\tilde{U}(0,t) = 0$. Therefore, $\tilde{U}$ solve the wave equation on the half-space.

Remember, that there were two cases given by d'Alembert's formula for the solution of the wave equation on the half line. In this case we must choose the case corresponding to $t \geq r$ i.e. the case for large time. This is since we drive $r \to 0^+$ and so $r \leq t$. Hence the representation solution is,

$$\tilde{U}(r,t) = \frac{1}{2}\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{H}(y)\, dy.$$

Then we have that,

$$\begin{aligned}
rU(r,t) &= \frac{1}{2}\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{H}(y)\, dy \\
U(r,t) &= \frac{1}{r}\left[\frac{1}{2}\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{H}(y)\, dy\right] \\
U(r,t) &= \frac{1}{2r}\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right) + \frac{1}{2r}\int_{r-t}^{r+t} \tilde{H}(y)\, dy \\
\fint_{\partial B(x,r)} u(y,t)\, dS(y) &= \frac{1}{2r}\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right) + \frac{1}{2r}\int_{r-t}^{r+t} \tilde{H}(y)\, dy.
\end{aligned}$$

Since the left hand side is the average over the ball and the right hand functions $\tilde{G}$ and $\tilde{H}$ are bounded i.e. have compact support. We may recover u by simply letting $r \to 0$. Hence we get that,

$$u(x,t) = \frac{\tilde{G}'(0+t) - \tilde{G}'(t-0)(-1)}{2} + \frac{\tilde{H}(t+0) - \tilde{H}(-1)}{2}.$$

Notice we have used l'Hospital's rule to evaluate the limit on the right hand side for the $\tilde{G}$ term and the fundamental theorem of calculus for the $\tilde{H}$ term.

$$u(x,t) = \tilde{G}'(t) + \tilde{H}(t).$$

We would like to write everything in terms of u, g, h.

$$\begin{aligned}
u(x,t) &= \tilde{G}'(t) + \tilde{H}(t) \\
&= \frac{d}{dt}\left(tG(t)\right) + tH(t) \\
&= \frac{d}{dt}\left(t \fint_{\partial B(x,t)} g(y)\, dS(y)\right) + t \fint_{\partial B(x,t)} h(y)\, dS(y) \\
&= \fint_{\partial B(x,t)} g(y)\, dS(y) + t\frac{\partial}{\partial t}\left(\fint_{\partial B(x,t)} g(y)\, dS(y)\right) + t \fint_{\partial B(x,t)} h(y)\, dS(y) \\
&= \fint_{\partial B(x,t)} g(y)\, dS(y) + \fint_{\partial B(x,t)} Dg(y)\cdot(y-x)\, dS(y) + t \fint_{\partial B(x,t)} h(y)\, dS(y) \\
&= \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)\, dS(y).
\end{aligned}$$

The last line is known as *Kirchhoff's formula* and is the solution to the IBVP (2.42).

2.9.4 Energy Methods

We now turn to discussing some energy methods for the wave equation. In particular we want to show; given the problem,

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in} & [0,L]^n \times (0,\infty) \\ u(x,0) = 0,\ u_t(x,0) = 0 \text{ on} & \mathbb{R}^3 \times \{t=0\} \\ u(0,t) = 0 = u(L,t) \text{ on} & \{x \in [0,L]^n\} \times (0,\infty). \end{cases} \tag{2.43}$$

The only solution is given by $u(x,t) \equiv 0$. Although this claim may seem intuitive, it is actually highly non-trivial to prove mathematically. As it turns out, to see this claim most easily, we will need to use energy methods.

To gain a energy functional, the idea is to multiply by some clever functional and integrate over the space. For the heat and Laplace equation, we chose the test function to be u itself. However for the wave equation we choose u_t to be the test function. This will become clear shortly. Let the space domain be given as Ω, testing the PDE against u_t we gain,

$$u_{tt}u_t = \Delta u u_t \implies \int_\Omega u_{tt}u_t \, d\Omega = \int_\Omega \Delta u u_t \, d\Omega.$$

Consider the integrand on the left-hand side, we get

$$u_{tt}u_t = \frac{1}{2}\partial_t(u_t)^2 \implies \int_\Omega \frac{1}{2}\partial_t(u_t)^2 \, d\Omega = \partial_t \frac{1}{2}\left(\int_\Omega (u_t)^2 \, d\Omega\right).$$

Then, looking at the right-hand side, upon an integration by parts,

$$\int_\Omega \Delta u u_t \, d\Omega = \int_{\partial\Omega} u_t \cdot Du - \int_\Omega Du \cdot Du_t \, d\Omega = -\int_\Omega Du \cdot Du_t \, d\Omega = -\frac{d}{dt}\frac{1}{2}\int_\Omega |Du|^2 \, d\Omega.$$

the penultimate equality follows since the boundary integral must vanish because we know $u(0,t) = 0$ so $u_t(0,t) = 0$. Equating both sides we get,

$$\frac{d}{dt}\frac{1}{2}\left(\int_\Omega (u_t)^2 \, d\Omega\right) = -\frac{d}{dt}\frac{1}{2}\int_\Omega |Du|^2 \, d\Omega.$$

Writing everything in homogenous form,

$$\frac{d}{dt}\left[\frac{1}{2}\int_\Omega (u_t)^2 + |Du|^2 \, d\Omega\right] = 0.$$

If we now define the space integral as our energy functional,

$$E(t) := \frac{1}{2}\int_\Omega (u_t)^2 + |Du|^2 \, d\Omega,$$

then by the pervious equation, it must be the case that $E(t)$ is steady in time. Physically, this is clearly a conservation of energy statement. What this also tells us is that $E(0) = E(t)$. Hence if the initial data is zero at time zero, then indeed the solution is identically zero.

Based on uniqueness results via the energy method, we can also see that we may apply the same idea to show uniqueness of the wave equation Cauchy problem. Simply by taking u, v as two solutions to the problem and defining the new function $w := u - v$ and using the analysis above to show $w \equiv 0$.

3 Revisiting the Transport Equation: motivating Weak Solutions to PDEs

We now depart from the general layout and treatment as provided by Evans' PDE. Following the structure of the course; Prof. Slim decided to a give a brilliant lecture motivating the idea of weak solutions to PDEs by examining once again the transport equation. We pursue the same here.

Recall the transport equation Cauchy problem,

$$\begin{cases} u_t - cu_x &= 0 \\ u(x,0) &= u_0(x). \end{cases} \tag{3.1}$$

For simplicity, we take $n = 1$. We know the solution to this problem takes the form $u(x,t) = u_0(x - ct)$. Recall also that for the existence of solutions we require the initial data to have some regularity, namely, $u_0 \in C^1$.

Upon a moments reflection, this requirement may seem too strong. Consider the physical meaning of the solution; it is simply that some wave (or bump) moves to the right or left at some velocity c. But then this should also be true for some continuous not differentiable function such as the bump. Futhermore, this behaviour should also be reflected for a discontinuous function for example the step-function. As such, it seems $u_0 \in C^1$ may be too restrictive. The question then becomes, how do we reconcile this regularity assumption and the fact that it is possible to find discontinuous functions that can exhibit the same behaviour as the solutions of differential equations (in this case the transport PDE).

Answering this question will yield the deep albeit intuitive notion of ***weak solutions***[7] to PDEs.

Take $u \in C^1(\mathbb{R} \times (0,\infty))$ to be a solution to (3.1) and take a *test function*, $\varphi \in C_c^1(\mathbb{R} \times [0,\infty))$. Notice φ has compact support in both space and time. Multiplying the PDE by the test function and integrating over space-time. We do this term by term and integrate by parts.

$$\int_0^\infty \int_{\mathbb{R}} u_x(x,t)\varphi(x,t)\,dx\,dt = \int_0^\infty [u\varphi]_{x=-R}^{x=R}\,dt - \int_0^\infty \int_{\mathbb{R}} u\varphi_x\,dx\,dt = -\int_0^\infty \int_{\mathbb{R}} u\varphi_x\,dx\,dt$$

$$\int_0^\infty \int_{\mathbb{R}} u_t(x,t)\varphi(x,t)\,dx\,dt = \int_{\mathbb{R}} [u\varphi]_{t=0}^{t=\infty}\,dx - \int_0^\infty \int_{\mathbb{R}} u\varphi_t\,dx\,dt = -\int_{\mathbb{R}} u_0(x)\varphi(x,0)\,dx - \int_0^\infty \int_{\mathbb{R}} u\varphi_t\,dx\,dt$$

Hence, we may write the integral version of the transport PDE,

$$\int_0^\infty \int_{\mathbb{R}} u(x,t)\left[\varphi_t(x,t) + c\varphi_x(x,t)\right]\,dx\,dt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\,dx = 0. \tag{3.2}$$

This form of the PDE is valid for any $\varphi \in C_c^1$ in space-time. So it follows that the only regularity condition we need to impose on u is local integrability.

The key idea behind weak solutions is that we now seek solutions u of (3.2) for all φ. This is not in general the same as solutions to the original PDE. Notice then, we have weakened the restrictions on u to be a solution. Now it need not even be differentiable or even continuous! Yet is still is a "weak" solution to the integral equation.

Definition 3.1. ***Weak Solution***: *For $u_0 \in L^1_{Loc}(\mathbb{R})$, we say that u is a weak solution to the Cauchy problem if,*

- *u is locally integrable in space-time.*
- *$\forall\, \varphi \in C^1(\mathbb{R} \times [0,\infty))$, (3.2) is satisfied.*

[7] Weak solutions are also referred to as generalized solutions or solutions in the sense of distributions.

We now make some very useful remarks that we will refer back to implicitly when doing calculations and writing proofs.

If $u \in C^1(\mathbb{R} \times (0,\infty))$ is a classical solution to the transport PDE, then u is also a weak solution. This is a deep claim and will require proof that uses fundamental measure and functional analytic arguments. We will develop all this shortly.

We say a weak solution belongs to the set $D'(\mathbb{R}\times(0,\infty))$; borrowing from the notation in distribution theory where D' is the dual space of the set of distributions; that is the set of smooth functions with compact support in space-time.

Let $u \in L^1_{\text{Loc}}(\mathbb{R} \times (0,\infty))$, then,

$$\int_{\mathbb{R}} u\varphi = 0 \iff u(x,t) \equiv 0 \text{ a.e.}, \tag{3.3}$$

for all $\varphi \in C_c^\infty$. The above is also true if $u \in L^1(\mathbb{R})$.

Proposition 3.2. *For the transport PDE and IVP* (3.1), *a weak solution is a strong solution.*

Proof. From regularity that has already been discussed regarding the transport equation, we know that $u \in C^1$. Furthermore, from (3.2) we also know how to represent the PDE in terms of test functions $\varphi \in C_c^1$. So we want to show that u is a classical solution.

Fix $\varphi \in C_c^\infty(\mathbb{R} \times (0,\infty))$. Notice, fixing this test function implies the space integral concerning u_0 will disappear from (3.2). Hence upon performing an integration by parts, using the fact the φ has compact support to make the boundary term vanish, we get

$$\int_0^\infty \int_{\mathbb{R}} (u_t + cu_x)\varphi \, dx \, dt = 0.$$

But this equation is true if and only if $u_t + xu_x = 0$ a.e. which is obviously the transport PDE.

Now we must ensure the initial conditions are recovered. Setting $t = 0$, and performing the IBP,

$$\int_{\mathbb{R}} ((u(x,0) - u_0(x)))\varphi(x,0) \, dx = 0.$$

Suppose we chose the test function such that $\varphi(x,t) = \psi(x)\Theta(t)$ with $\Theta(0) = 1$. Then,

$$\int_{\mathbb{R}} ((u(x,0) - u_0(x)))\psi(x) \, dx = 0.$$

which can only be true iff $(u(x,0) - u_0(x)) \equiv 0$ a.e. The claim follows. ☐

Theorem 3.3. ***Existence of Weak Solutions:*** *Let $u_0 \in L^1_{Loc}(\mathbb{R})$ the $u(x,t) = u_0(x-ct)$ is a weak solution to $\Box u = 0$. Moreover the map $t \mapsto u(x,t) = u_0(x-ct)$ is continuous.*

To be clear, here continuity is meant in terms of sequential continuity; for $t_n \to t$ in $\mathbb{R}$ and any compact $K \subset \mathbb{R}$,

$$\int_K |u(x,t_n) - u(x,t)| \to 0, \text{ as } n \to \infty.$$

Proof. Take $\varphi \in C_c^1(\mathbb{R} \times [0,\infty))$, we want to show

$$\int_0^\infty \int_{\mathbb{R}} u_0(x-ct)(\varphi_t + c\varphi_x) \, dx \, dt + \int_{\mathbb{R}} u_0(x)\varphi(x,0) \, dx = 0.$$

Let $y = x - ct$, then the integral form of the PDE transforms as

$$\int_R u(y) \int_0^\infty \frac{d}{dt}\varphi(y+ct)\, dt\, dy + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\, dx = -\int_{\mathbb{R}} u_0(y)\varphi(u,0)\, dy + \int_{\mathbb{R}} u_0(y)\varphi(u,0)\, dy = 0.$$

□

We now turn to proving uniqueness of weak solutions. As we have seen, constructing weak solutions is far easier than constructing classical solutions. However, we will find difficulty is always conserved as hence, proving uniqueness of weak solutions will be harder than proving uniqueness of classical solutions. This is true in general and hence will be true for the linear transport PDE as well.

Theorem 3.4. ***Uniqueness of Weak Solutions.***

Proof. Let u_1 and u_2 be two weak solutions. If $u := u_1 - u_2$, we want to show $u \equiv 0$. Taking this as a change of variables, we want to solve the IVP,

$$\begin{cases} u_t + cu_x &= 0 \\ u(x,0) &= 0. \end{cases} \tag{3.4}$$

Note, (3.4) follows from the integral version of the PDE written in terms of the two weak solutions.

Now, for any $\varphi \in C_c^\infty(\mathbb{R} \times [0,\infty))$, we have that,

$$\int_0^\infty \int_{\mathbb{R}} (\varphi_t + c\varphi_x) u(x,t)\, dx\, dt = 0.$$

The initial condition term disappears since $u_0 = 0$. To show $u \equiv 0$, it is enough to show,

$$\int_0^\infty \int_{\mathbb{R}} \psi(x,t) u(x,t)\, dx\, dt = 0,$$

for all test functions ψ. So fix $\psi \in C_c^\infty$ and consider the dual problem,

$$\begin{cases} u_t + cu_x &= \psi(x,t),\ x \in \mathbb{R},\ t > 0 \\ \varphi(x,T) &= 0. \end{cases} \tag{3.5}$$

We recognize this IVP to simply be the non-homogenous transport IVP with homogenous initial data. Hence the representation solution is given by,

$$\varphi(x,t) = \int_T^t \psi(s, x - (t-s)c)\, ds.$$

We know that ψ is smooth with compact support but we must be able to prove the same for φ.

If $t \geq T$, then $s \in [T,t]$, by compact support of ψ and the initial data, it must be the case that $\psi = 0$. Which implies $\varphi = 0$ for all $x \in \mathbb{R}$. Hence φ has compact support in time.

We now want to show φ has compact support in space. If $|x| \geq R + |cT|$ then $|x - c(t-s)| \geq |x| - c|t-s| \geq |x| - cT \geq R$. Which again implies $\psi = 0$. Hence, $\varphi = 0$.

In summary we have that,

$$\int_0^\infty \int_{\mathbb{R}} \psi(x,t) u(x,t)\, dx\, dt = \int_0^\infty \int_{\mathbb{R}} (\varphi_t + c\varphi_x) u(x,t)\, dx\, dt = 0.$$

Hence it must be the case that $u = u_1 - u_2 = 0$. We are done. □

3.0.1 Properties of Weak Solutions:

- Max principles hold: If $u_0 \geq 0$, then $u(x,t) \geq 0$ a.e.
- Local energy decay in time. Given $K \subset \mathbb{R}$ is compact then,

$$\int_K |u| = \int_K |u_0| = \int_{K+ct} |u(y)|\, dy \to 0, \text{ as } t \to \infty.$$

- Global energy constant in time.

$$\int_{\mathbb{R}} |u| = \int_{\mathbb{R}} |u(x-ct)| = \int_{\mathbb{R}} |u(y)|\, dy = \text{const.}.$$

- Averaging effect show algebraic decay.

3.1 Non-Linear Transport Equation

Consider the non-linear transport PDE IVP,

$$\begin{cases} u_t + a(u)u_x = 0 \\ u(x,0) = u_0(x). \end{cases} \tag{3.6}$$

The equation is technically a quasi-linear PDE. We may also deduce the speed of solutions is variable due to the non-linearity brought on by $a(u)$.

We assume $a(u) \in C^1(\mathbb{R})$. With this assumption at hand, it will turn out expressing $a(u)$ in terms of its anti-derivative will be advantageous. So the IVP may be re-written as,

$$\begin{cases} u_t + \partial_x(f(u)) = 0 \\ u(x,0) = u_0(x). \end{cases}$$

We recover $a(u)$ via the chain rule; $\partial_x(f(u)) = f'(u)u_x = a(u)u_x$. The primary reason for doing this is since in n dimensions, we have written the PDE in a divergence form. Upon applying the divergence theorem we will be able to derive a conservation law.

3.1.1 Method of Characteristics: finding classical solutions of (3.6)

We now undertake the task of finding classical solutions of (3.6) using the Method of Characteristics that we briefly saw when finding representation solutions for the linear transport equation.

We begin by assuming there exists a solution $u \in C^1$ for (3.6). Then, the characteristic curves are given by the ODE system,

$$\begin{cases} \frac{dx}{dt} &= a(u(x(t))) \\ x(0) &= 0. \end{cases} \tag{3.7}$$

Since the PDE is non-linear the ODE for the curves depends on the solution u. However, from existence and uniqueness theory from ODEs; from the Cauchy-Lipschitz theorem, we are guaranteed the existence of unique solutions to (3.7).

By the chain rule we have that,

$$\frac{dx}{dt} = \dot{x} = u_x\dot{x} + u_t = u_x a(u) - u_x a(u) = 0,$$

by the non-linear transport PDE. Hence it must be the case that u is constant along each characteristic curve. Furthermore, from the initial condition, $u(x(0),0) = u(x_0,0) = u_0(x_0)$. So we have that $x(t) = x_0 + t(a(u_0(x_0)))$, therefore $u(x_0 + ta(u_0(x_0)), t) = u_0(x_0)$ for all $x_0 \in \mathbb{R}$.

Theorem 3.5. ***Local Existence of Classical Solutions:*** *Assume $u_0 \in C^1$ is bounded, assume also u_0' is also bounded. Let $T > 0$ be so that ... Then (3.6) has a unique solution on $[0, T]$ in $C^1(\mathbb{R} \times [0, T])$.*

Proof. We write the proof in terms of the result of a the following lemma.

Lemma 3.6. *Let $u_0 \in C^1(\mathbb{R})$ be bounded and u' be bounded and let $T > 0$. Then the map ϕ,*

$$(x, t) \mapsto (x + ta(u_0(x)), t),$$

is a diffeomorphism i.e. the map is a bijection and is differentiable and invertible.

Hence, we need only prove the map is differentiable and is one to one. Let $\psi_t(x) = x + ta(u_0(x))$. Then, for any $t \in [0, T]$,

$$\begin{aligned}\psi_t'(x) &= 1 + ta'(u_0)u_0'(x) \\ &\geq 1 + t \inf \left[a'(u_0)u_0'(x)\right].\end{aligned}$$

From this we may deduce ψ is a monotonically increasing function and hence must have an inverse. We must also show it is a bijection i.e. surjective and injective.

To show injection, consider $\psi(x, t) = \psi(y, s)$ i.e $a(u_0)t + x = a(u_0(y))s + y$ and $t = s$. Hence $\psi_t(x) = \psi_t(y)$ and so $x = y$.

To show surjection, let $(y, s) \in \mathbb{R} \times [0, T]$. Set $t = s$ and $x = \psi_t^{-1}(y)$. This also show the mapping is a bijection.

Lastly to show diffeomorphism the Jacobian must be non-singular. Take

$$|D\phi(x, t)| = a'(u_0(s))tu_0' + 1 > 0.$$

Hence ϕ is a C^1 diffeomorphism.

Returning now to the main proof. □

The big take away point here is that the non-linear transport equation will admit smooth solution but for only a short time on the interval $[0, T]$.

3.1.2 Conservation Laws and Shock Solutions

We now look at scalar conservation laws in one space dimension and how they relate to the non-linear transport equation. Consider the Cauchy problem,

$$\begin{cases} u_t + F(u)_x = 0 & \text{on } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{3.8}$$

From the previous section we know that (3.8) does not have a general smooth solution for all time. Hence it is wise to use the notion of weak solutions to find other, perhaps non-unique solutions.

We define a weak solution of (3.8) as follows. Let $\varphi \in C_c^\infty$ be a test function. Multiplying φ through the PDE and integrating over space-time,

$$0 = \int_{\mathbb{R}^+} \int_{\mathbb{R}} (u_t + F(u)_x)\varphi \, dx \, dt = -\int_{\mathbb{R}^+} \int_{\mathbb{R}} u\varphi_t \, dx \, dt - \left[\int_{\mathbb{R}} u\varphi \, dx\right]_{t=0} \int_{\mathbb{R}^+} \int_{\mathbb{R}} F(u)\varphi_x \, dx \, dt.$$

Cleaning up the equation we may re-write the PDE in terms of integrals over the test function,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} u\varphi_t + F(u)\varphi_x \, dx\, dt + \left[\int_{\mathbb{R}} g\varphi \, dx\right]_{t=0} = 0. \tag{3.9}$$

So, we may say u is a weak solution of (3.8) if it obeys (3.9) for all test functions φ.

The question then becomes; given a weak solution of (3.8), what can we deduce about behaviour of solutions. Let us answer this question by fixing a weak solution with a simple structure.

In particular, let $V \subset \mathbb{R} \times (0, \infty)$ and let u be smooth on either side of a smooth curve C in V. Let C induce the partition on V where V_l and V_r are regions on the left and right hand side of C respectively. We also assume u is a weak solution but u has uniformly continuous derivatives in both V_l and V_r.

Pick $\varphi \in C_c^\infty(V_l)$. Integrating (3.9) by parts we get that,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} u\varphi_t + F(u)\varphi_x \, dx\, dt = -\int_{\mathbb{R}^+}\int_{\mathbb{R}} (u_t + F(u)_x)\varphi \, dx\, dt = 0.$$

Which is only true if $u_t + F(u)_x = 0$ a.e. The same argument holds for V_r.

If we now pick a test function with compact support in V that need not vanish on C, we may express the integral equation as a decomposition over V_r and V_l,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} u\varphi_t + F(u)\varphi_x \, dx\, dt = \iint_{V_l} u\varphi_t + F(u)\varphi_x \, dx\, dt + \iint_{V_r} u\varphi_t + F(u)\varphi_x \, dx\, dt.$$

Take the integral over V_l and integrate by parts,

$$\iint_{V_l} u\varphi_t + F(u)\varphi_x \, dx\, dt = -\iint_{V_l} [u_t + F(u)_x]\varphi \, dx\, dt + \int_C (u_l\nu_2 + F(u_l)\nu_1)\varphi \, dl = \int_C (u_l\nu_2 + F(u_l)\nu_1)\varphi \, dl.$$

Here we take (ν_1, ν_2) as the unit normal to the curve C and u_l denotes the left hand limit. We may use the same argument for V_r. If we then add these identities for the right and left we find that,

$$\int_C [F(u_l) - F(u_r)\nu_1 + (u_l - u_r)\nu_2]\varphi \, dl = 0.$$

Since this equality must hold for all test functions it must be the case that,

$$F(u_l) - F(u_r)\nu_1 + (u_l - u_r)\nu_2 = 0, \text{ along } C.$$

Suppose then C is represented parametrically. More precisely, take C to be a characteristic curve of the solution, $\{(x,t) | x = s(t)\}$ for some smooth $s(\cdot) : [0, \infty) \to \mathbb{R}$. We may take $(\nu_1, \nu_2) = (1 + \dot{s}^2)^{-1/2}(1, -\dot{s})$. So we get that $F(u_l) - F(u_r) = \dot{s}(u_l - u_r)$, in V along C.

We may re-write this as,

$$[[F(u)]] = \sigma[[u]]. \tag{3.10}$$

This is known as the ***Rankin-Hugoniot Condition***. Physically, they tell us the speed of solutions across a discontinuity curve.

Example 3.7. *Consider the example of shock wave creation in the case of a Cauchy problem regarding Burger's equation.*

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{on } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{3.11}$$

Where the initial data is given by,

$$g = \begin{cases} 1 & if\, x \leq 0 \\ 1-x & if\, x \leq 0 \\ 0 & if\, x \geq 1. \end{cases}$$

From the method of characteristics, we know that transport equations and non-linear transport equations will simply propagate the intial data through characteristic curves. As such the solution will be given by $z = g(x_0)$ via along characteristic curves,

$$y(s) = (g(x_0)s + x_0, s),\ \forall x_0 \in \mathbb{R}.$$

4 An Interlude on Functional Analysis: Fourier Transform and Sobolev Spaces

We now define the Fourier transform and then use this to introduce the foundation of modern PDE analysis, the ***Sobolev Spaces***.

4.1 The Fourier Transform

We introduce the Fourier transform by first defining the Schwartz class of functions $S(\mathbb{R}^n)$ and the tempered distributions. For brevity we skip explicit proofs of the propositions and theorems as these can be found in any measure theory/analysis textbook. See for example, Real Analysis by Folland.

We define $S(\mathbb{R}^n)$ to be the set of smooth functions that decay at infinity faster than any polynomial. In particular, for all multi-indices α, β, $C_{\alpha,\beta} > 0$,

$$|x^\alpha \partial_x^\beta u| \leq C_{\alpha,\beta}.$$

A ready consequence of the definition of S is the embedding $C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$. This is since S contains the set of all smooth functions that decay faster at infinity than any polynomial that may or may not have compact support. Hence, the set of compactly supported smooth functions must necessarily be a proper subspace of S. For example, consider the Gaussian, $f(x) = \exp(-|x|^2)$, then $f \in S$ but $f \notin C_c^\infty$.

We define the Fourier Transform in $\mathbb{R}^n$ for $u \in S$ as,

$$\mathcal{F}(u) = \hat{u} := \int_{\mathbb{R}^n} \exp(-i\xi \cdot x) u(x)\, dx. \tag{4.1}$$

The inverse Fourier Transform is given as,

$$\mathcal{F}^{-1}(u) := \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \exp(i\xi \cdot x) u(\xi)\, d\xi. \tag{4.2}$$

We also have the following theorem.

Theorem 4.1. *The Fourier Transform is a bijection between $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$, with inverse $\mathcal{F}^{-1}$.*

We also have the following well known properties,

- $\mathcal{F}(\partial_j u) = i\xi_j \mathcal{F}(u)$.
- $F(x_j u) = i\partial_{\xi_j} \mathcal{F}(u)$.

In words, the Fourier Transform maps differentiation in real space to multiplication by $i\xi$ and vice-versa.

Theorem 4.2. ***Plancheral:*** *For $u, v \in S$,*

$$\int_{\mathbb{R}^n} \mathcal{F}(u) F(v)\, d\xi = \int_{\mathbb{R}^n} u F(v)\, dx.$$

In particular, if we take $u = v$, we arrive at the all important Parseval identity,

$$\int |u|^2 \, dx = 2\pi^n \int |u|^2 \, d\xi. \tag{4.3}$$

Hence we see, that on the space of square integrable functions i.e. L^2, the Fourier Transform is an isometry.

We now turn to discussing tempered distributions and how the connect to the Fourier Transform.

Definition 4.3. ***Tempered Distribution:*** *We denote $S'(\mathbb{R}^n)$ to be the set of linear, continuous maps $T : S(\mathbb{R}^n) \to \mathbb{C}$. We call T a tempered distribution.*

In particular, let $f \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. We denote the tempered distribution associated with f by,

$$T_f := \int_{\mathbb{R}_n} f(x)\varphi(x) \, dx,$$

where $\varphi \in S$. Indeed T is linear and continuous and so $T_f \in S'$.

We now turn to defining the Fourier Transform for tempered distributions.

Definition 4.4. *Let $T \in S'(\mathbb{R}^n)$, then the Fourier Transform of T is denoted $\mathcal{F}(T)$ and defined in the same was as before.*

Of course, here the subtle point is that T is a tempered distribution and is therefore in itself a linear and continuous mapping. Hence, since the distribution will involve a test function $\varphi \in S$, we have that $\mathcal{F}(T)(\varphi) = T(\mathcal{F}(\varphi))$. This identity will hold via an integration by parts. The IBP is in itself valid since we are dealing with smooth functions that do not see infinity.

Hence we may express these ideas in terms of duality brackets,

$$\langle \mathcal{F}(T), \varphi \rangle = \langle T, \mathcal{F}(\varphi) \rangle = \int_{\mathbb{R}^n} T(\xi)\mathcal{F}(\varphi)(\xi) \, d\xi = \int_{\mathbb{R}^n} \mathcal{F}(T)(x)\varphi(x) \, dx.$$

4.2 Sobolev Spaces